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565      **Supplementary Material: Parametric estimation of  $P(X > Y)$  for normal**  
distributions in the context of probabilistic environmental risk assessment

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723			
724			

## 725 1 R-code

726 This section contains the R codes used to perform the simulation. It consists of the main simulation program  
727 and the user-defined functions used in the main program.

### 728 1.1 Functions

#### 729 1.1.1 pnorm1

730 Function to determine scale of estimator - original or probit

```
731 pnorm1 <- function(x){  
732     #original  
733     #newR = pnorm(x)  
734     #probit  
735     newR = x  
736     newR  
737 }
```

#### 738 1.1.2 mode\_f

739 Function to estimate mode

```
740 mode_f <- function(x) {  
741     dens = try(density(x),silent=TRUE)  
742     if (class(dens)=="try-error") mode=0 else  
743         mode = dens$x[which.max(dens$y)]  
744     if (mode<0) mode=0 #only to use on on original scale  
745     mode  
746 }
```

#### 748 1.1.3 R\_MLE\_f

749 Function that calculates the MLE of  $P(X > Y)$  for the normal-normal model

```
750 R_MLE_f <- function(x,y,c){  
751     if (class(x) == "matrix") {  
752         nx = nrow(x)  
753         ny = nrow(y)  
754         m1 = apply(x,2,mean)  
755         m2 = apply(y,2,mean)  
756         var1 = apply(x,2,var)  
757         var2 = apply(y,2,var)  
758     } else {  
759         nx = length(x)  
760         ny = length(y)  
761         m1 = mean(x)  
762         m2 = mean(y)  
763         var1 = var(x)  
764         var2 = var(y)  
765     }  
766     theta = (m1 - m2 - c)/ sqrt(var1/nx*(nx-1) + var2/ny*(ny-1))  
767     R_MLE = pnorm1(theta)  
768     R_MLE  
769 }
```

#### 770 1.1.4 R\_QMLE\_f

771 Function that calculates the QMLE of  $P(X > Y)$  for the normal-normal model

```
772 R_QMLE_f <- function(x,y,c){  
773     if (class(x) == "matrix") {  
774         m1 = apply(x,2,mean)  
775         m2 = apply(y,2,mean)  
776         var1 = apply(x,2,var)  
777         var2 = apply(y,2,var)  
778     } else {  
779         m1 = mean(x)  
780         m2 = mean(y)
```

```

781     var1 = var(x)
782     var2 = var(y)
783   }
784   theta = (m1 - m2 - c)/ sqrt(var1 + var2)
785   R_QMLE = pnorm1(theta)
786   R_QMLE
787 }
```

### 1.1.5 R\_NP\_f

Function that calculates the non-parametric estimator of  $P(X > Y)$

```

790 R_NP_f <- function(exp,sen,c){
791   nx = length(exp)
792   ny = length(sen)
793   vec = rep(NA,length(c))
794   for (i in 1:length(c)) {
795     x_y = outer(exp,(sen+c[i]), "-")
796     gt=sum(x_y>1e-15)
797     eq = sum(abs(x_y)< 1e-15) # ties
798     r = (gt + 0.5*eq + 1)/(nx*ny + 2)
799     vec[i] = qnorm(r)
800   }
801   vec
802 }
```

### 1.1.6 comp\_sampler\_f

Function that generates sample of size  $n$  from posterior distribution of  $R$  and calculates the HPD credible intervals of posterior sample

```

803 library(coda)
804 comp_sampler_f <- function (n, seed, xbar, ybar, n_x, n_y, a_x, a_y, b_x, b_y,c,p2,p1) {
805   B = length(xbar)
806   samples = matrix(NA,nrow = n, ncol = B)
807   int2 = matrix(NA,nrow = B, ncol = 2)
808   int1 = matrix(NA,nrow = B, ncol = 1)
809   set.seed(seed)
810   for (i in 1:B) {
811     bx = b_x[i]
812     by = b_y[i]
813     xmean = xbar[i]
814     ymean = ybar[i]
815     s2x = 1/rgamma(n, shape = a_x, rate = bx)
816     s2y = 1/rgamma(n, shape = a_y, rate = by)
817     mx = rnorm(n, xmean, sqrt(s2x/n_x))
818     my = rnorm(n, ymean, sqrt(s2y/n_y))
819     r = pnorm1((mx - my - c) / sqrt(s2x + s2y))
820     samples[,i] = r
821     #calculate 2-sided HPD interval
822     int2[i,] = HPDinterval(mcmc(r),prob = p2)
823     #calculate 1-sided percentile interval
824     int1[i] = quantile(r,probs=p1)
825   }
826   return(list(samples = samples,intervals2 = int2,intervals1 = int1))
827 }
828 }
```

### 1.1.7 R\_boot\_f

Function to pass to statistic in boot to manually generate parametric bootstrap for  $R$

```

829 R_boot_f <- function(x,i,nx,ny,m1,m2,var1,var2,c,func){
830   exp_new = rnorm(nx,m1,sqrt(var1))
831   sen_new = rnorm(ny,m2,sqrt(var2))
832   R = func(exp_new,sen_new,c)
833   R
834 }
```

```
839 1.1.8 R_bootstrap_f
```

```
840 Function to pass to statistic in boot to generate non-parametric bootstrap for R
```

```
841 R_bootstrap_f <- function(data,i,func,nx,ny,c){  
842   newdata = data[i]  
843   exp = newdata[1:nx]  
844   sen = newdata[(nx+1):(nx+ny)]  
845   r = func(exp,sen,c)  
846   r  
847 }
```

```
848 1.2 Main Program
```

```
849 library(boot)  
850 library(coda)  
851  
852 #version of code  
853 vers = "v10"  
854 #sigma case  
855 factor = 1  
856  
857 #Parameter choices  
858 n_x = rep(c(5,100),each=5)  
859 n_y = rep(c(2,5,12,20,100),2)  
860 mu_x = -2.948  
861 mu_y = -2.948  
862 sig2_x = 0.0738  
863 sig2_y = (factor*sqrt(0.0738))^2  
864  
865 n_scen = length(n_x)  
866 B = 4500 #number of simulations  
867 n_boot = 1000 #number of bootstrap iterations  
868  
869 p2 = 0.9 #confidence level for 2-sided CI  
870 p1 = 0.95 #confidence level for 1-sided CI  
871  
872 #mean difference producing range of R values  
873 if (factor == 1)  
874   c = seq(0,2.9,0.1) else  
875   if (factor == 5)  
876     c = seq(0,10.4,0.4) else  
877     if (factor == 0.2)  
878       c = seq(0,2.1,0.1)  
879  
880 R_true =pnorm1((mu_x - mu_y - c)/sqrt(sig2_x + sig2_y))  
881  
882 #starting seed list for MC simulations for each scenario  
883 set.seed(1,kind="Mersenne-Twister")  
884 seed_list1 = sample(4:1e8,n_scen)  
885  
886 #starting seed list for bootstraps for each scenarios  
887 set.seed(2,kind="Mersenne-Twister")  
888 seed_list2 = sample(4:1e8,n_scen)  
889  
890 #starting seed list for MCMC for each scenario  
891 set.seed(3,kind="Mersenne-Twister")  
892 seed_list3 = sample(4:1e8,n_scen)  
893  
894 #-----  
895 # Generating B exposure and B sensitivity samples for each scenario  
896 #-----  
897  
898 for (scen in 1:10){  
899   set.seed(seed_list1[scen],kind = "Mersenne-Twister") #set a random seed per scenario  
900   #Parameter values for scen'th scenario
```

```

902     nx = n_x[scen]
903     ny = n_y[scen]
904
905     #Generating and writing samples
906     MC_x = replicate(B,rnorm(nx,mu_x,sqrt(sig2_x)))
907     name_x = paste("output_",vers,"/exposure/Scenario_",scen,
908                     "_Exposure.RData",sep="")
909     save(MC_x,file = name_x)
910
911     MC_y = replicate(B,rnorm(ny,mu_y,sqrt(sig2_y)))
912     name_y = paste("output_",vers,"/sensitivity_",factor,"/Scenario_",scen,
913                     "_Sensitivity.RData",sep="")
914     save(MC_y,file = name_y)
915 }
916
917 #-----
918 #             MLE estimator
919 #-----
920
921 bounds = list()
922 data = rep(1,10) #fake data vector for boot function
923
924 for (scen in 1:n_scen){
925     set.seed(seed_list2[scen],kind = "Mersenne-Twister")
926     exp_name = paste("output_",vers,"/exposure/Scenario_",scen,"_Exposure.RData",sep="")
927     sen_name = paste("output_",vers,"/sensitivity_",factor,"/Scenario_",scen,
928                     "_Sensitivity.RData",sep="")
929     load(exp_name, .GlobalEnv)
930     load(sen_name, .GlobalEnv)
931     nx = nrow(MC_x)
932     ny = nrow(MC_y)
933     num = rep(0,length(c))
934     R_MLE = matrix(NA,nrow = length(c),ncol = B)
935     l_MLE2 = matrix(NA,nrow = length(c),ncol = B)
936     u_MLE2 = matrix(NA,nrow = length(c),ncol = B)
937     u_MLE1 = matrix(NA,nrow = length(c),ncol = B)
938
939     #loop for confidence intervals
940     for (sim in 1:B) {
941         m1 = mean(MC_x[,sim])
942         m2 = mean(MC_y[,sim])
943         var1 = var(MC_x[,sim])
944         var2 = var(MC_y[,sim])
945         R_MLE[,sim] = pnorm1((m1 - m2 - c)/ sqrt(var1/nx*(nx-1) + var2/ny*(ny-1)))
946         s <- function(x,Mhat,deltahat,fhat,prob){
947             arg = sqrt(Mhat) * deltaxhat
948             nc = sqrt(Mhat) * x
949             pt(arg,df=fhat,ncp=nc) - prob
950         }
951         Mhat = (var1 + var2)/(var1/nx + var2/ny)
952         fhat = (var1 + var2)^2/(var1^2/(nx-1) + var2^2/(ny-1))
953         deltaxhat = (m1 - m2 - c)/(sqrt(var1/nx*(nx-1) + var2/ny*(ny-1)))
954         for (i in 1:length(c)) {
955             l_MLE2[i,sim] = uniroot.all(s,interval=c(-500,500),Mhat = Mhat,fhat = fhat,
956                                         deltaxhat = deltaxhat[i],prob = p1)
957             u_MLE2[i,sim] = uniroot.all(s,interval=c(-500,500),Mhat = Mhat,fhat = fhat,
958                                         deltaxhat = deltaxhat[i],prob = 1-p1)
959         }
960     }
961     u_MLE1 = u_MLE2
962     bca = B-num #number of intervals where BCa was used
963     bounds[[scen]] = list(l_MLE2,u_MLE2,u_MLE1,R_MLE,bca)
964 }
965 name = paste("output_",vers,"/estimators/MLE_raw_output_",factor,".RData",sep="")
966 save(bounds,file = name)

```

```

968 #-----
969 #          QuasiMLE estimator
970 #-----
971
972 bounds = list()
973 data = rep(1,10) #fake data vector for boot function
974
975 for (scen in 1:n_scen){
976   set.seed(seed_list2[scen],kind = "Mersenne-Twister")
977   exp_name = paste("output_",vers,"/exposure/Scenario_",scen,"_Exposure.RData",sep="")
978   sen_name = paste("output_",vers,"/sensitivity_",factor,"/Scenario_",scen,
979                   "_Sensitivity.RData",sep="")
980   load(exp_name, .GlobalEnv)
981   load(sen_name, .GlobalEnv)
982   nx = nrow(MC_x)
983   ny = nrow(MC_y)
984   num = rep(0,length(c))
985   R_QMLE = matrix(NA,nrow = length(c),ncol = B)
986   l_QMLE2 = matrix(NA,nrow = length(c),ncol = B)
987   u_QMLE2 = matrix(NA,nrow = length(c),ncol = B)
988   u_QMLE1 = matrix(NA,nrow = length(c),ncol = B)
989
990   #loop for confidence intervals
991   for (sim in 1:B) {
992     m1 = mean(MC_x[,sim])
993     m2 = mean(MC_y[,sim])
994     var1 = var(MC_x[,sim])
995     var2 = var(MC_y[,sim])
996     R_QMLE[,sim] = pnorm1((m1 - m2 - c)/ sqrt(var1 + var2))
997     s <- function(x,Mhat,deltahat,fhat,prob){
998       arg = sqrt(Mhat) * deltaxhat
999       nc = sqrt(Mhat) * x
1000      pt(arg,df=fhat,ncp=nc) - prob
1001    }
1002    Mhat = (var1 + var2)/(var1/nx + var2/ny)
1003    fhat = (var1 + var2)^2/(var1^2/(nx-1) + var2^2/(ny-1))
1004    deltaxhat = (m1 - m2 - c)/(sqrt(var1 + var2))
1005    for (i in 1:length(c)) {
1006      l_QMLE2[i,sim] = uniroot.all(s,interval=c(-500,500),Mhat = Mhat,fhat = fhat,
1007                                     deltaxhat = deltaxhat[i],prob = p1)
1008      u_QMLE2[i,sim] = uniroot.all(s,interval=c(-500,500),Mhat = Mhat,fhat = fhat,
1009                                     deltaxhat = deltaxhat[i],prob = 1-p1)
1010    }
1011  }
1012  u_QMLE1 = u_QMLE2
1013  bca = B-num #number of intervals where BCa was used
1014  bounds[[scen]] = list(l_QMLE2,u_QMLE2,u_QMLE1,R_QMLE,bca)
1015 }
1016 name = paste("output_",vers,"/estimators/QMLE_raw_output_",factor,".RData",sep="")
1017 save(bounds,file = name)
1018
1019 #-----
1020 #          Bayes estimator
1021 #-----
1022
1023 n_bayes = 1000 #posterior sample size
1024 bounds = list()
1025
1026 for (scen in 1:n_scen){
1027   seed = seed_list3[scen]
1028   exp_name = paste("output_",vers,"/exposure/Scenario_",scen,"_Exposure.RData",sep="")
1029   sen_name = paste("output_",vers,"/sensitivity_",factor,"/Scenario_",scen,
1030                   "_Sensitivity.RData",sep="")
1031   load(exp_name, .GlobalEnv)
1032   load(sen_name, .GlobalEnv)
1033 }
```

```

1034 #sample parameters estimator
1035 nx = nrow(MC_x)
1036 ny = nrow(MC_y)
1037 m_x = apply(MC_x,2,mean)
1038 m_y = apply(MC_y,2,mean)
1039 var_x = apply(MC_x,2,var)
1040 var_y = apply(MC_y,2,var)
1041
1042 #Bayes posterior parameters
1043 a_x = 0.5*(nx-1)
1044 a_y = 0.5*(ny-1)
1045 b_x = 0.5*(nx-1)*var_x
1046 b_y = 0.5*(ny-1)*var_y
1047
1048 R_Bayes_mean = matrix(NA,nrow = length(c),ncol = B)
1049 R_Bayes_mode = matrix(NA,nrow = length(c),ncol = B)
1050 R_Bayes_median = matrix(NA,nrow = length(c),ncol = B)
1051 l_Bayes2 = matrix(NA,nrow = length(c),ncol = B)
1052 u_Bayes2 = matrix(NA,nrow = length(c),ncol = B)
1053 u_Bayes1 = matrix(NA,nrow = length(c),ncol = B)
1054 for (i in 1:length(c)){
1055   object = comp_sampler_f(n_bayes,seed,m_x, m_y , nx, ny, a_x, a_y, b_x, b_y,c[i], p2,p1)
1056   s = object$samples
1057   Post_median = apply(s,2,median)
1058   Post_mean = apply(s,2,mean)
1059   Post_mode = apply(s,2,mode_f)
1060   int2 = object$intervals2
1061   int1 = object$intervals1
1062   l_Bayes2[i,] = int2[,1]
1063   u_Bayes2[i,] = int2[,2]
1064   u_Bayes1[i,] = int1
1065   R_Bayes_median[i,] = Post_median
1066   R_Bayes_mode[i,] = Post_mode
1067   R_Bayes_mean[i,] = Post_mean
1068 }
1069 bounds[[scen]] = list(l_Bayes2,u_Bayes2,u_Bayes1,
1070                       R_Bayes_median,R_Bayes_mode,R_Bayes_mean)
1071 }
1072 name = paste("output_",vers,"/estimators/Bayes_raw_output_",factor,".RData",sep="")
1073 save(bounds,file = name)
1074
1075 #-----
1076 #      Nonparametric estimator
1077 #-----
1078
1079 bounds = list()
1080
1081 for (scen in 1:n_scen){
1082   set.seed(seed_list2[scen],kind = "Mersenne-Twister")
1083   exp_name = paste("output_",vers,"/exposure/Scenario_",scen,"_Exposure.RData",sep="")
1084   sen_name = paste("output_",vers,"/sensitivity_",factor,"/Scenario_",scen,
1085                     "_Sensitivity.RData",sep="")
1086   load(exp_name, .GlobalEnv)
1087   load(sen_name, .GlobalEnv)
1088   nx = nrow(MC_x)
1089   ny = nrow(MC_y)
1090   R_NP = matrix(NA,nrow = length(c),ncol = B)
1091   l_NP2 = matrix(NA,nrow = length(c),ncol = B)
1092   u_NP2 = matrix(NA,nrow = length(c),ncol = B)
1093   u_NP1 = matrix(NA,nrow = length(c),ncol = B)
1094   num = rep(0,length(c))
1095
1096 #loop for bootstrapping
1097   for (sim in 1:B) {
1098     d = c(MC_x[,sim],MC_y[,sim])
1099     myboot = boot(d,R_bootstrap_f,R=n_boot,strata = c(rep(1,nx),rep(2,ny)),

```

```

1100         func = R_NP_f ,nx=nx,ny=ny,c = c)
1101 R_NP[,sim] = myboot$t0
1102 for (i in 1:length(c)){
1103     int = try(boot.ci(myboot,conf = p2,type="bca",index=i)$bca[,4:5],silent = TRUE)
1104     if (!is.numeric(int)) {
1105         int = try(boot.ci(myboot,conf = p2,type="perc",index=i)$percent[,4:5],silent = TRUE)
1106         num[i] = num[i] + 1
1107     }
1108     if (!is.numeric(int))
1109         int = c(myboot$t0[i],myboot$t0[i])
1110     if (is.na(int[2])) int[2] = max(myboot$t[,i])
1111     if (is.na(int[1])) int[1] = min(myboot$t[,i])
1112     l_NP2[i,sim] = int[1]
1113     u_NP2[i,sim] = int[2]
1114 }
1115 }
1116 u_NP1 = u_NP2
1117 bca = B - num
1118 bounds[[scen]] = list(l_NP2,u_NP2,u_NP1,R_NP,bca)
1119 }
1120 name = paste("output_",vers,"/estimators/NP_raw_output_",factor,".RData",sep="")
1121 save(bounds,file = name)

```

## 1.3 Figures

### 1.3.1 Coverage Figures

This code produces the coverage figures for the 2-sided intervals. The coverage figures for the upper bounds are produced in a similar manner.

```

1126 library(boot) #bootstrap
1127 library(coda) #for HPD intervals
1128 library(xtable) #to convert table to latex code
1129 library(ggplot2) #to use ggplot
1130 library(gridExtra) #for facet spacing
1131 #-----
1132 # file name components
1133 #-----
1134 #version of code
1135 vers = "v10"
1136 #sigma case
1137 factor = 1
1138
1139 #Parameter choices
1140 n_x = rep(c(5,100),each=5)
1141 n_y = rep(c(2,5,12,20,100),2)
1142 mu_x = -2.948
1143 mu_y = -2.948
1144 sig2_x = 0.0738
1145 sig2_y = (factor*sqrt(sig2_x))^2
1146
1147 n_scen = length(n_x)
1148 source("functions/functions.R")
1149
1150 B = 4500 #number of simulations to be performed
1151
1152 #choosing the scale
1153 pnorm1 <- function(x){
1154     #original
1155     #newR = pnorm(x)
1156     #probit
1157     newR = x
1158     newR
1159 }
1160
1161 #creating labels for facet graph
1162 mf_labeller <- function(var, value){

```

```

1163     value <- as.character(value)
1164     if (var=="nx") {
1165       value[value==5] <- "\n Exposure sample size: 5\n"
1166       value[value==100]   <- "\n Exposure sample size: 100\n"
1167     }
1168     if (var=="Estimator") {
1169       value[value=="QMLE_t"] <- "\n QMLE - noncentral t\n"
1170       value[value=="QMLE_boot"]   <- "\n QMLE - bootstrap\n"
1171       value[value=="Bayes"] <- "\n Bayesian\n"
1172       value[value=="NP"]   <- "\n Non-parametric\n"
1173     }
1174     return(value)
1175   }
1176 
1177 p2 = 0.9 #confidence level for 2-sided CI
1178 p1 = 0.95 #confidence level for 1-sided CI
1179 
1180 if (factor == 1)
1181   #for equal variances
1182   c = seq(0,2.9,0.1) else #mean difference producing range of R values
1183   if (factor == 5)
1184     #for 5x
1185     c = seq(0,10.4,0.4) else #mean difference producing range of R values
1186     if (factor == 0.2)
1187       #for 0.2x
1188       c = seq(0,2.1,0.1) #mean difference producing range of R values
1189 
1190 R_true = pnorm1((mu_x - mu_y - c)/sqrt(sig2_x + sig2_y))
1191 Rmatrix = matrix(R_true,nrow = length(R_true),ncol = B)
1192 
1193 #-----
1194 #  QMLE - noncentral t
1195 #-----
1196 
1197 name = paste("output_",vers,"/estimators/QMLE_raw_output_",factor,"_noncent",
1198             ".RData",sep="")
1199 load(name)
1200 
1201 #Coverage and lengths
1202 coverage = matrix(NA,nrow=length(R_true),ncol=n_scen)
1203 lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1204 std_lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1205 interval_data = NULL
1206 for (scen in 1:n_scen){
1207   lower = bounds[[scen]][[1]]
1208   upper = bounds[[scen]][[2]]
1209   count = apply((Rmatrix>(lower) & Rmatrix<(upper)),1,sum)
1210   coverage[,scen] = count/B
1211   lengths[,scen] = apply((upper - lower),1,median)
1212   std_lengths[,scen] = lengths[,scen]/abs(R_true)
1213   std_lengths[1,scen] = std_lengths[2,scen]
1214   #matrix of n_scenxlength(c) by 6 containing coverage, std_median_lengths
1215   #number of bca intervals, nx and ny
1216   interval_data = rbind(interval_data,cbind(coverage[,scen],std_lengths[,scen],
1217                                         pnorm(R_true),bounds[[scen]][[5]],
1218                                         rep(n_x[scen],length(c)),
1219                                         rep(n_y[scen],length(c))))
1220 }
1221 coverage_data_QMLE_t = data.frame(interval_data,rep("QMLE_t",length(c)))
1222 colnames(coverage_data_QMLE_t) = c("Coverage", "Med_length","R_true","Bca","nx","ny","Estimator")
1223 coverage_data_QMLE_t$nx = factor(coverage_data_QMLE_t$nx)
1224 coverage_data_QMLE_t$ny = factor(coverage_data_QMLE_t$ny)
1225 
1226 #-----
1227 #  QMLE - bootstrap
1228 #-----

```

```

1229
1230 name = paste("output_",vers,"/estimators/QMLE_raw_output_",factor,".RData",sep="")
1231 load(name)
1232
1233 #Coverage and lengths
1234 coverage = matrix(NA,nrow=length(R_true),ncol=n_scen)
1235 lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1236 std_lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1237 interval_data = NULL
1238 for (scen in 1:n_scen){
1239   lower = bounds[[scen]][[1]]
1240   upper = bounds[[scen]][[2]]
1241   count = apply((Rmatrix>(lower) & Rmatrix<(upper)),1,sum)
1242   coverage[,scen] = count/B
1243   lengths[,scen] = apply((upper - lower),1,median)
1244   std_lengths[,scen] = lengths[,scen]/abs(R_true)
1245   std_lengths[1,scen] = std_lengths[2,scen]
1246   #matrix of n_scenxlength(c) by 6 containing coverage, std_median_lengths
1247   #number of bca intervals, nx and ny
1248   interval_data = rbind(interval_data,cbind(coverage[,scen],std_lengths[,scen],
1249                                         pnorm(R_true),bounds[[scen]][[5]],
1250                                         rep(n_x[scen],length(c)),
1251                                         rep(n_y[scen],length(c))))
1252 }
1253 coverage_data_QMLE_boot = data.frame(interval_data,rep("QMLE_boot",length(c)))
1254 colnames(coverage_data_QMLE_boot) = c("Coverage", "Med_length","R_true","Bca","nx","ny","Estimator")
1255 coverage_data_QMLE_boot$nx = factor(coverage_data_QMLE_boot$nx)
1256 coverage_data_QMLE_boot$ny = factor(coverage_data_QMLE_boot$ny)
1257
1258 #-----
1259 # Bayes
1260 #-----
1261
1262 name = paste("output_",vers,"/estimators/Bayes_raw_output_",factor,
1263 ".RData",sep="")
1264 load(name)
1265
1266 #Coverage and lengths
1267 coverage = matrix(NA,nrow=length(R_true),ncol=n_scen)
1268 lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1269 std_lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1270 interval_data = NULL
1271 for (scen in 1:n_scen){
1272   lower = bounds[[scen]][[1]]
1273   upper = bounds[[scen]][[2]]
1274   count = apply((Rmatrix>(lower) & Rmatrix<(upper)),1,sum)
1275   coverage[,scen] = count/B
1276   lengths[,scen] = apply((upper - lower),1,median)
1277   std_lengths[,scen] = lengths[,scen]/abs(R_true)
1278   std_lengths[1,scen] = std_lengths[2,scen]
1279   #matrix of n_scenxlength(c) by 6 containing coverage, std_median_lengths
1280   #number of bca intervals, nx and ny
1281   interval_data = rbind(interval_data,cbind(coverage[,scen],std_lengths[,scen],
1282                                         pnorm(R_true),rep(NA,length(c)),
1283                                         rep(n_x[scen],length(c)),
1284                                         rep(n_y[scen],length(c))))
1285 }
1286 coverage_data_Bayes = data.frame(interval_data,rep("Bayes",length(c)))
1287 colnames(coverage_data_Bayes) = c("Coverage", "Med_length","R_true","Bca","nx","ny","Estimator")
1288 coverage_data_Bayes$nx = factor(coverage_data_Bayes$nx)
1289 coverage_data_Bayes$ny = factor(coverage_data_Bayes$ny)
1290
1291 #-----
1292 # NP - Laplace
1293 #-----

```

```

1295 name = paste("output_",vers,"/estimators/NP_raw_output_",factor,
1296 ".RData",sep="")
1297 load(name)
1298
1299 #Coverage and lengths
1300 coverage = matrix(NA,nrow=length(R_true),ncol=n_scen)
1301 lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1302 std_lengths = matrix(NA,nrow=length(R_true),ncol=n_scen)
1303 interval_data = NULL
1304 for (scen in 1:n_scen){
1305   lower = bounds[[scen]][[1]]
1306   upper = bounds[[scen]][[2]]
1307   count = apply((Rmatrix>(lower) & Rmatrix<(upper)),1,sum)
1308   coverage[,scen] = count/B
1309   lengths[,scen] = apply((upper - lower),1,median)
1310   std_lengths[,scen] = lengths[,scen]/abs(R_true)
1311   std_lengths[1,scen] = std_lengths[2,scen]
1312   #matrix of n_scenxlength(c) by 6 containing coverage, std_median_lengths
1313   #number of bca intervals, nx and ny
1314   interval_data = rbind(interval_data,cbind(coverage[,scen],std_lengths[,scen],
1315                           pnorm(R_true),bounds[[scen]][[5]],
1316                           rep(n_x[scen],length(c)),
1317                           rep(n_y[scen],length(c))))
1318 }
1319 coverage_data_NP = data.frame(interval_data,rep("NP",length(c)))
1320 colnames(coverage_data_NP) = c("Coverage", "Med_length","R_true","Bca","nx","ny","Estimator")
1321 coverage_data_NP$nx = factor(coverage_data_NP$nx)
1322 coverage_data_NP$ny = factor(coverage_data_NP$ny)
1323 coverage_data_NP$Coverage[coverage_data_NP$Coverage<0.5]=0.5
1324
1325 min(coverage_data_NP$Coverage)
1326
1327 alldata = rbind(coverage_data_QMLE_boot,coverage_data_QMLE_t,
1328                   coverage_data_Bayes,coverage_data_NP)
1329 alldata_12 = subset(alldata,ny==12)
1330
1331 #-----
1332 # graph
1333 #-----
1334
1335 #plot for coverages vs med CI length
1336 g = ggplot(data=alldata) +
1337   geom_point(aes(Coverage,Med_length,colour=R_true,size=ny),shape=16) +
1338   geom_point(data=alldata_12,aes(Coverage,Med_length),colour="black",size=5,shape=1) +
1339   #geom_line(aes(group=ny),colour = "black") +
1340   geom_vline(xintercept=0.9) +
1341   geom_hline(yintercept=0, linetype="dashed") +
1342   theme_bw() +
1343   #ggtitle("Scatterplot of coverage vs median CI length") +
1344   ylab("Relative Median Interval Length") + xlab("Coverage") +
1345   theme(panel.grid.major = element_blank(),
1346         panel.grid.minor = element_blank(),
1347         panel.background = element_blank(),
1348         panel.margin = unit(1,"cm"),
1349         strip.text.x = element_text(size=18),
1350         strip.text.y = element_text(size=18),
1351         axis.title.x = element_text(size=22,vjust=-0.8),
1352         axis.title.y = element_text(size=22,vjust=0.3),
1353         axis.text.x = element_text(size=18),
1354         axis.text.y = element_text(size=18),
1355         plot.title = element_text(size=22,vjust=2),
1356         legend.title = element_text(size=20),
1357         legend.text = element_text(size=18),
1358         legend.position = "bottom",
1359         legend.key.width = unit(1,"line"),
1360         legend.key = element_rect(linetype=0)) +

```

```

1361 guides(color = guide_colourbar(barwidth=25)) +
1362   scale_colour_gradientn(name="True R: ",colours=rainbow(4),
1363     breaks=c(0,0.1,0.2,0.3,0.4,0.5)) +
1364   scale_size_manual(name="Effect sample size: ",
1365     breaks = c(2,5,12,20,100),
1366     values = c(3,4,5,6,7),
1367     labels = c("2    ","5    ","12   ","20   ","100")) +
1368   scale_y_continuous(breaks = seq(0,12,3),
1369     expand=c(0,0),limits=c(-1,12)) +
1370   scale_x_continuous(breaks=seq(0.5,1,by=0.1),expand=c(0,0),limits=c(0.5,1)) +
1371   facet_grid(Estimator~nx,labeller=mf_labeller)
1372
1373 name = paste("coverages_lengths_2s_",factor,".png",sep="")
1374 ggsave(file=name,plot = g,width=10,height=18)
1375 ggsave(file="test_noncent2.png",plot = g,width=10,height=18)

1376 1.3.2 Sampling distribution of estimators

1377 #version of code
1378 vers = "v9"
1379 #sigma case
1380 factor = 5
1381
1382 n_x = rep(c(5,100),each=5)
1383 n_y = rep(c(2,5,12,20,100),2)
1384 mu_x = -2.948
1385 mu_y = -2.948
1386 sig2_x = 0.0738
1387 sig2_y = (factor*sqrt(sig2_x))^2
1388
1389 n_scen = length(n_x)
1390
1391 pnorm1 <- function(x){
1392   #original
1393   #newR = pnorm(x)
1394   #probit
1395   newR = x
1396   newR
1397 }
1398
1399 source("functions/functions.R")
1400
1401 if (factor == 1)
1402   #for equal variances
1403   c = seq(0,2.9,0.1) else #mean difference producing range of R values
1404   if (factor == 5)
1405     #for 5x
1406     c = seq(0,10.4,0.4) else #mean difference producing range of R values
1407     if (factor == 0.2)
1408       #for 0.2x
1409       c = seq(0,2.1,0.1) #mean difference producing range of R values
1410
1411 R_true = pnorm1((mu_x - mu_y - c)/sqrt(sig2_x + sig2_y))
1412
1413 name = paste("output_",vers,"/estimators/MLE_raw_output_",factor,
1414   ".RData",sep=""))
1415 load(name)
1416 MLE_bounds = bounds
1417 name = paste("output_",vers,"/estimators/QMLE_raw_output_",factor,
1418   ".RData",sep=""))
1419 load(name)
1420 QMLE_bounds = bounds
1421 name = paste("output_",vers,"/estimators/Bayes_raw_output_",factor,
1422   ".RData",sep=""))
1423 load(name)
1424 Bayes_bounds = bounds

```

```

1425 name = paste("output_",vers,"/estimators/NP_raw_output_",factor,
1426 ".RData",sep="")
1427 load(name)
1428 NP_bounds = bounds
1429 name = paste("output_",vers,"/estimators/NP_raw_output_inf_",factor,
1430 ".RData",sep="")
1431 load(name)
1432 NPi_bounds = bounds
1433 #name = paste("output_",vers,"/estimators/NP_raw_output_ss_",factor,
1434 # ".RData",sep="")
1435 #load(name)
1436 #NPs_bounds = bounds
1437
1438 plots = list()
1439
1440 for (scen in 1:n_scen){
1441   R_MLE = (MLE_bounds[[scen]][[4]])
1442   R_QMLE = (QMLE_bounds[[scen]][[4]])
1443   R_Bayes = (Bayes_bounds[[scen]][[4]])
1444   R_NP = (NP_bounds[[scen]][[4]])
1445   R_NPi = (NPi_bounds[[scen]][[4]])
1446   R_NPs = (NPs_bounds[[scen]][[4]])
1447   l_MLE = apply(R_MLE,1,quantile,probs=0.025)
1448   l_QMLE = apply(R_QMLE,1,quantile,probs=0.025)
1449   l_Bayes = apply(R_Bayes,1,quantile,probs=0.025)
1450   l_NP = apply(R_NP,1,quantile,probs=0.025)
1451   l_NPi = apply(R_NPi,1,quantile,probs=0.025)
1452   l_NPs = apply(R_NPs,1,quantile,probs=0.025)
1453   u_MLE = apply(R_MLE,1,quantile,probs=0.975)
1454   u_QMLE = apply(R_QMLE,1,quantile,probs=0.975)
1455   u_Bayes = apply(R_Bayes,1,quantile,probs=0.975)
1456   u_NP = apply(R_NP,1,quantile,probs=0.975)
1457   u_NPi = apply(R_NPi,1,quantile,probs=0.975)
1458   u_NPs = apply(R_NPs,1,quantile,probs=0.975)
1459   m_MLE = apply(R_MLE,1,quantile,probs=0.5)
1460   m_QMLE = apply(R_QMLE,1,quantile,probs=0.5)
1461   m_Bayes = apply(R_Bayes,1,quantile,probs=0.5)
1462   m_NP = apply(R_NP,1,quantile,probs=0.5)
1463   m_NPi = apply(R_NPi,1,quantile,probs=0.5)
1464   m_NPs = apply(R_NPs,1,quantile,probs=0.5)
1465
1466 min_R = qnorm(rep(1/(n_x[scen]*n_y[scen]+1),length(c)))
1467
1468 data_MLE = data.frame(R=R_true,lb = min_R,lower=l_MLE,med=m_MLE,upper=u_MLE,
1469 est=rep("MLE",length(c)))
1470 data_QMLE = data.frame(R=R_true,lb = min_R,lower=l_QMLE,med=m_QMLE,upper=u_QMLE,
1471 est=rep("QMLE",length(c)))
1472 data_Bayes = data.frame(R=R_true,lb = min_R,lower=l_Bayes,med=m_Bayes,upper=u_Bayes,
1473 est=rep("Bayes",length(c)))
1474 data_NP = data.frame(R=R_true,lb = min_R,lower=l_NP,med=m_NP,upper=u_NP,
1475 est=rep("NP",length(c)))
1476
1477 totaldata = rbind(data_MLE,data_QMLE,data_Bayes,data_NP)
1478
1479 plot = ggplot(data=totaldata,aes(x=R,y=lower)) +
1480 geom_line(data=totaldata,aes(x=R,y=lower,colour=est,linetype=est),size=3) +
1481 #geom_line(data=totaldata,aes(x=R,y=med,colour=est,linetype=est),size=3) +
1482 geom_line(data=totaldata,aes(x=R,y=upper,colour=est,linetype=est),size=3) +
1483 geom_line(data=totaldata,aes(x=R,y=lb),size=1,colour="grey") +
1484 geom_abline(intercept = 0, slope = 1,linetype = "dotted") +
1485 theme_bw() +
1486 #annotate(geom = "text",label="(D)",x=2.75,y=0.5,size=7) +
1487 ggttitle(paste("nx=",n_x[scen]," ny=",n_y[scen],sep="")) +
1488 ylab(expression(probit(hat(R)))) + xlab(expression(probit(R))) +
1489 theme(panel.grid.major = element_blank(),
1490 panel.grid.minor = element_blank(),

```

```

1491     panel.background = element_blank(),
1492     axis.title.x = element_text(size=22),
1493     axis.title.y = element_text(size=22),
1494     axis.text.x = element_text(size=18),
1495     axis.text.y = element_text(size=18),
1496     plot.title = element_text(size=22,vjust=2),
1497     legend.title = element_text(size=22),
1498     legend.text = element_text(size=18),
1499     legend.position = "bottom",
1500     legend.key = element_rect(linetype=0),
1501     legend.key.width = unit(3,"cm"),
1502     plot.margin = unit(c(10,10,10,10),"mm")) +
1503     scale_y_continuous(breaks = c(-13,-11,-9,-7,-5,-3,-1,1),
1504                         expand=c(0,0),limits=c(-13,1.2)) +
1505     #scale_y_continuous(breaks = c(1e-14,1e-12,1e-10,1e-8,1e-6,1e-4,1e-2,1),
1506     #                         expand=c(0,0),
1507     #                         trans=log2_trans(),
1508     #                         labels = trans_format("log10", math_format(10^.x)),
1509     #                         limits=c(1e-14,1)) +
1510     scale_x_continuous(breaks = c(-7,-5,-3,-1,1),
1511                         expand=c(0,0),limits=c(-8,1.2)) +
1512     scale_linetype_manual(name="Estimator ",
1513                           values=c("solid","longdash","dashed","dotdash"),
1514                           labels=c("MLE ","QMLE ","Bayes ","NP ")) +
1515     scale_colour_manual(name="Estimator ",
1516                           values=c("#E69F00", "#009E73" , "#CC79A7","#56B4E9"),
1517                           labels=c("MLE ","QMLE ","Bayes ","NP "))

1518   plots[[scen]] <- plot
1519 }
1520 legend = g_legend_f(plot)
1521 figure = arrangeGrob(arrangeGrob(plots[[1]]+theme(legend.position="none"),
1522                               plots[[6]]+theme(legend.position="none"),
1523                               plots[[2]]+theme(legend.position="none"),
1524                               plots[[7]]+theme(legend.position="none"),
1525                               plots[[3]]+theme(legend.position="none"),
1526                               plots[[8]]+theme(legend.position="none"),
1527                               plots[[4]]+theme(legend.position="none"),
1528                               plots[[9]]+theme(legend.position="none"),
1529                               plots[[5]]+theme(legend.position="none"),
1530                               plots[[10]]+theme(legend.position="none"),
1531                               nrow=5),
1532                               #main=textGrob("2.5% and 97.5% percentiles of sampling distribution",
1533                               gp=gpar(fontsize=26))),
1534                               legend, nrow=2,heights=c(27,1))
1535 #name = paste("sampling_distribution_R_median_",factor,".png",sep="")
1536 name = paste("sampling_distribution_R_percentiles_",factor,".png",sep="")
1537 ggsave(file=name,plot = figure,width=14,height=28)

1538
1539 1.3.3 Bayes posterior summary statistics
1540 setwd("D:/PhD/Thesis/Paper_2/R_Files")
1541
1542 library(coda) #for HPD intervals
1543 library(ggplot2)
1544 library(gridExtra) #for legend specifications
1545 library(scales) #log scale in ggplot
1546 library(modeest) #to calculate mode
1547
1548 #-----
1549 # file name components
1550 #-----
1551 #version of code
1552 vers = "v9"
1553 #sigma case
1554 factor = 5

```

```

1555
1556 #Parameter choices
1557 n_x = rep(c(5,100),each=5)
1558 n_y = rep(c(2,5,12,20,100),2)
1559 mu_x = -2.948
1560 mu_y = -2.948
1561 sig2_x = 0.0738
1562 sig2_y = (factor*sqrt(sig2_x))^2
1563
1564 n_scen = length(n_x)
1565
1566 source("functions/functions.R")
1567 pnorm1 <- function(x){
1568   #original
1569   #newR = pnorm(x)
1570   #probit
1571   newR = x
1572   newR
1573 }
1574
1575 if (factor == 1)
1576   #for equal variances
1577   c = seq(0,2.9,0.1) else #mean difference producing range of R values
1578   if (factor == 5)
1579     #for 5x
1580     c = seq(0,10.4,0.4) else #mean difference producing range of R values
1581     if (factor == 0.2)
1582       #for 0.2x
1583       c = seq(0,2.1,0.1) #mean difference producing range of R values
1584
1585 R_true = pnorm1((mu_x - mu_y - c)/sqrt(sig2_x + sig2_y))
1586
1587 #Only scenario 8
1588 scen=8
1589 #Original
1590
1591 R_true = pnorm((mu_x - mu_y - c)/sqrt(sig2_x + sig2_y))
1592
1593 #Study difference in posterior summary statistics
1594 plots = list()
1595
1596 name = paste("output_",vers,"/estimators/Bayes_raw_output_orig",factor,
1597 ".Rdata",sep="")
1598 load(name)
1599
1600 R_mean = bounds[[scen]][[6]]
1601 R_median = bounds[[scen]][[4]]
1602 R_mode = bounds[[scen]][[5]]
1603 l_mean = apply(R_mean,1,quantile,probs=0.025)
1604 u_mean = apply(R_mean,1,quantile,probs=0.975)
1605 m1_mean = apply(R_mean,1,quantile,probs=0.5)
1606 l_median = apply(R_median,1,quantile,probs=0.025)
1607 u_median = apply(R_median,1,quantile,probs=0.975)
1608 m1_median = apply(R_median,1,quantile,probs=0.5)
1609 l_mode = apply(R_mode,1,quantile,probs=0.025)
1610 u_mode = apply(R_mode,1,quantile,probs=0.975)
1611 m1_mode = apply(R_mode,1,quantile,probs=0.5)
1612
1613 data_mean = data.frame(R=R_true,lower=l_mean,med=m1_mean,upper=u_mean,
1614 est=rep("Mean ",length(c)))
1615 data_median = data.frame(R=R_true,lower=l_median,med=m1_median,upper=u_median,
1616 est=rep("Median ",length(c)))
1617 data_mode = data.frame(R=R_true,lower=l_mode,med=m1_mode,upper=u_mode,
1618 est=rep("Mode ",length(c)))
1619
1620 totaldata = rbind(data_median,data_mean,data_mode)

```

```

1621
1622 plot1 = ggplot(data=totaldata,aes(x=R,y=lower,group=est,colour=est,linetype=est)) +
1623   #geom_line(data=totaldata,aes(x=R,y=lower),size=1.5) +
1624   geom_line(data=totaldata,aes(x=R,y=med,group=est),size=3) +
1625   #geom_line(data=totaldata,aes(x=R,y=upper),size=1.5) +
1626   geom_abline(intercept = 0, slope = 1,linetype="dotted") +
1627   theme_bw() +
1628   annotate(geom = "text",label="(A)",x=1e-13,y=0.2,size=7) +
1629   ylab(expression(hat(R))) + xlab(expression(R)) +
1630   theme(panel.grid.major = element_blank(),
1631         panel.grid.minor = element_blank(),
1632         panel.background = element_blank(),
1633         axis.title.x = element_text(size=22),
1634         axis.title.y = element_text(size=22),
1635         axis.text.x = element_text(size=18),
1636         axis.text.y = element_text(size=18),
1637         plot.title = element_text(size=22,vjust=2),
1638         legend.title = element_text(size=22),
1639         legend.text = element_text(size=18),
1640         legend.position = "bottom",
1641         legend.key = element_rect(linetype=0),
1642         legend.key.width = unit(3,"cm"),
1643         plot.margin = unit(c(10,10,10,10),"mm")) +
1644   scale_y_continuous(breaks = c(1e-14,1e-12,1e-10,1e-8,1e-6,1e-4,1e-2,1),
1645                     expand=c(0,0),
1646                     trans=log2_trans(),
1647                     labels = trans_format("log10", math_format(10^.x)),
1648                     limits=c(1e-14,1)) +
1649   scale_x_continuous(breaks = c(1e-14,1e-12,1e-10,1e-8,1e-6,1e-4,1e-2,1),
1650                     expand=c(0,0),
1651                     trans=log2_trans(),
1652                     labels = trans_format("log10", math_format(10^.x)),
1653                     limits=c(1e-14,1)) +
1654   scale_linetype_manual(name="Estimator ",
1655                         values=c("solid","dotdash","dashed")) +
1656   scale_colour_manual(name="Estimator ",values=c("#E69F00", "#CC79A7", "#56B4E9"))

1657
1658
1659 plot2 = ggplot(data=totaldata,aes(x=R,y=lower,group=est,colour=est,linetype=est)) +
1660   geom_line(data=totaldata,aes(x=R,y=lower),size=3) +
1661   #geom_line(data=totaldata,aes(x=R,y=med,group=est),size=2) +
1662   geom_line(data=totaldata,aes(x=R,y=upper),size=3) +
1663   geom_abline(intercept = 0, slope = 1,linetype="dotted") +
1664   theme_bw() +
1665   annotate(geom = "text",label="(B)",x=1e-13,y=0.2,size=7) +
1666   ylab(expression(hat(R))) + xlab(expression(R)) +
1667   theme(panel.grid.major = element_blank(),
1668         panel.grid.minor = element_blank(),
1669         panel.background = element_blank(),
1670         axis.title.x = element_text(size=22),
1671         axis.title.y = element_text(size=22),
1672         axis.text.x = element_text(size=18),
1673         axis.text.y = element_text(size=18),
1674         plot.title = element_text(size=22,vjust=2),
1675         legend.title = element_text(size=22),
1676         legend.text = element_text(size=18),
1677         legend.position = "bottom",
1678         legend.key = element_rect(linetype=0),
1679         legend.key.width = unit(3,"cm"),
1680         plot.margin = unit(c(10,10,10,10),"mm")) +
1681   scale_y_continuous(breaks = c(1e-14,1e-12,1e-10,1e-8,1e-6,1e-4,1e-2,1),
1682                     expand=c(0,0),
1683                     trans=log2_trans(),
1684                     labels = trans_format("log10", math_format(10^.x)),
1685                     limits=c(1e-14,1)) +
1686   scale_x_continuous(breaks = c(1e-14,1e-12,1e-10,1e-8,1e-6,1e-4,1e-2,1),

```

```

1687         expand=c(0,0),
1688         trans=log2_trans(),
1689         labels = trans_format("log10", math_format(10^x)),
1690         limits=c(1e-14,1)) +
1691     scale_linetype_manual(name="Estimator ",
1692                           values=c("solid","dotdash","dashed")) +
1693     scale_colour_manual(name="Estimator ",values=c("#E69F00", "#CC79A7", "#56B4E9"))
1694
1695
1696 #probit
1697 R_true = pnorm1((mu_x - mu_y - c)/sqrt(sig2_x + sig2_y))
1698 name = paste("output_",vers,"/estimators/Bayes_raw_output_",factor,
1699               ".Rdata",sep="")
1700 load(name)
1701
1702 R_mean = bounds[[scen]][[6]]
1703 R_median = bounds[[scen]][[4]]
1704 R_mode = bounds[[scen]][[5]]
1705 l_mean = apply(R_mean,1,quantile,probs=0.025)
1706 u_mean = apply(R_mean,1,quantile,probs=0.975)
1707 m1_mean = apply(R_mean,1,quantile,probs=0.5)
1708 l_median = apply(R_median,1,quantile,probs=0.025)
1709 u_median = apply(R_median,1,quantile,probs=0.975)
1710 m1_median = apply(R_median,1,quantile,probs=0.5)
1711 l_mode = apply(R_mode,1,quantile,probs=0.025)
1712 u_mode = apply(R_mode,1,quantile,probs=0.975)
1713 m1_mode = apply(R_mode,1,quantile,probs=0.5)
1714
1715 data_mean = data.frame(R=R_true,lower=l_mean,med=m1_mean,upper=u_mean,
1716   est=rep("Mean    ",length(c)))
1717 data_median = data.frame(R=R_true,lower=l_median,med=m1_median,upper=u_median,
1718   est=rep("Median   ",length(c)))
1719 data_mode = data.frame(R=R_true,lower=l_mode,med=m1_mode,upper=u_mode,
1720   est=rep("Mode    ",length(c)))
1721
1722 totaldata = rbind(data_median,data_mean,data_mode)
1723
1724 plot3 = ggplot(data=totaldata,aes(x=R,y=lower,group=est,colour=est,linetype=est)) +
1725   #geom_line(data=totaldata,aes(x=R,y=lower),size=1.5) +
1726   geom_line(data=totaldata,aes(x=R,y=med,group=est),size=3) +
1727   #geom_line(data=totaldata,aes(x=R,y=upper),size=1.5) +
1728   geom_abline(intercept = 0, slope = 1,linetype="dotted") +
1729   theme_bw() +
1730   annotate(geom = "text",label="(C)",x=-8.2,y=0.2,size=7) +
1731   ylab(expression(probit(hat(R)))) + xlab(expression(probit(R))) +
1732   theme(panel.grid.major = element_blank(),
1733         panel.grid.minor = element_blank(),
1734         panel.background = element_blank(),
1735         axis.title.x = element_text(size=22),
1736         axis.title.y = element_text(size=22),
1737         axis.text.x = element_text(size=18),
1738         axis.text.y = element_text(size=18),
1739         plot.title = element_text(size=22,vjust=2),
1740         legend.title = element_text(size=22),
1741         legend.text = element_text(size=18),
1742         legend.position = "bottom",
1743         legend.key = element_rect(linetype=0),
1744         legend.key.width = unit(3,"cm"),
1745         plot.margin = unit(c(10,10,10,10),"mm")) +
1746   scale_y_continuous(breaks = c(-11,-9,-7,-5,-3,-1,1),
1747                      expand=c(0,0),limits=c(-13,1)) +
1748   scale_x_continuous(breaks = c(-9,-7,-5,-3,-1,1),
1749                      expand=c(0,0),limits=c(-9,1)) +
1750   scale_linetype_manual(name="Estimator ",
1751                         values=c("solid","dotdash","dashed")) +
1752   scale_colour_manual(name="Estimator ",values=c("#E69F00", "#CC79A7", "#56B4E9"))

```

```

1753
1754
1755 plot4 = ggplot(data=totaldata,aes(x=R,y=lower,group=est,colour=est,linetype=est)) +
1756   geom_line(data=totaldata,aes(x=R,y=lower),size=3) +
1757   #geom_line(data=totaldata,aes(x=R,y=med,group=est),size=2) +
1758   geom_line(data=totaldata,aes(x=R,y=upper),size=3) +
1759   geom_abline(intercept = 0, slope = 1,linetype="dotted") +
1760   theme_bw() +
1761   annotate(geom = "text",label="(D)",x=-8.2,y=0.2,size=7) +
1762   ylab(expression(probit(hat(R)))) + xlab(expression(probit(R))) +
1763   theme(panel.grid.major = element_blank(),
1764         panel.grid.minor = element_blank(),
1765         panel.background = element_blank(),
1766         axis.title.x = element_text(size=22),
1767         axis.title.y = element_text(size=22),
1768         axis.text.x = element_text(size=18),
1769         axis.text.y = element_text(size=18),
1770         plot.title = element_text(size=22,vjust=2),
1771         legend.title = element_text(size=22),
1772         legend.text = element_text(size=18),
1773         legend.position = "bottom",
1774         legend.key = element_rect(linetype=0),
1775         legend.key.width = unit(3,"cm"),
1776         plot.margin = unit(c(10,10,10,10),"mm")) +
1777   scale_y_continuous(breaks = c(-11,-9,-7,-5,-3,-1,1),
1778                      expand=c(0,0),limits=c(-13,1)) +
1779   scale_x_continuous(breaks = c(-9,-7,-5,-3,-1,1),
1780                      expand=c(0,0),limits=c(-9,1)) +
1781   scale_linetype_manual(name="Estimator ",
1782                         values=c("solid","dotdash","dashed")) +
1783   scale_colour_manual(name="Estimator ",values=c("#E69F00", "#CC79A7", "#56B4E9"))

1784
1785 legend = g_legend_f(plot1)
1786 figure_sub = arrangeGrob(arrangeGrob(plot1+theme(legend.position="none"),
1787                                   plot2+theme(legend.position="none"),
1788                                   plot3+theme(legend.position="none"),
1789                                   plot4+theme(legend.position="none"),
1790                                   nrow=2),
1791                                   legend, nrow=2,heights=c(14,1))
1792 ggsave(file="Bayes_posterior_summary_sub.png",plot = figure_sub,width=14,height=15)
1793

```

## 2 Figures and Tables

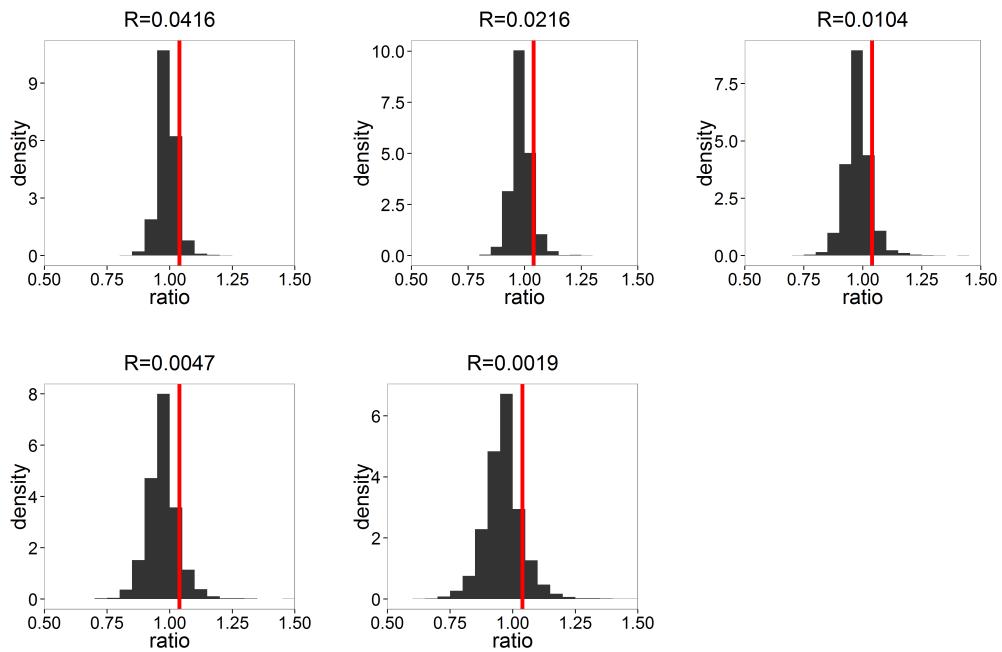
### 2.1 Application

In this section, we provide the extra tables and figures relating to the case study.

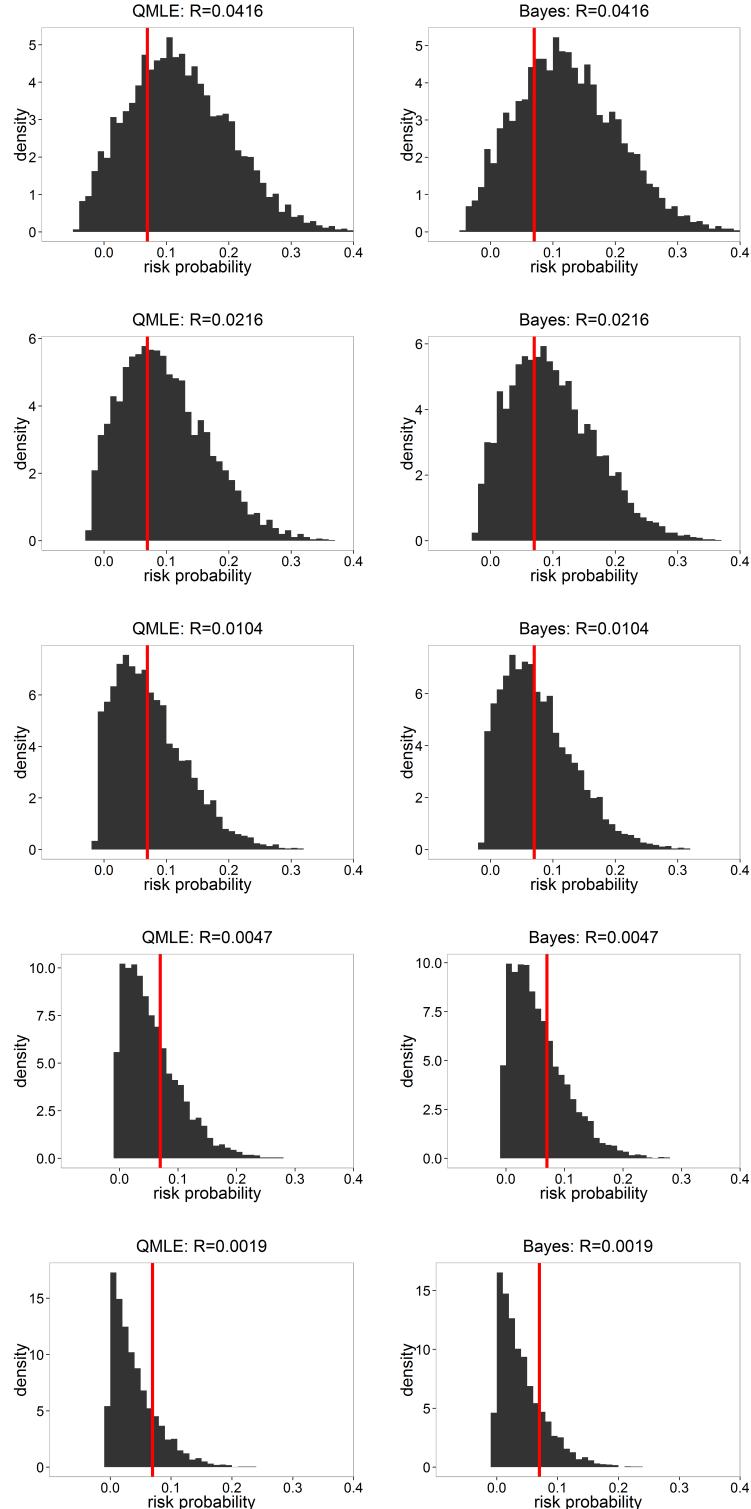
**Table S1:** Test statistic values and p-values for six normality tests on the nano-Ag exposure and effect concentration data of Gottschalk et al. (2013).

Normality Test	Exposure	Effect
	Statistic (p-value)	Statistic (p-value)
Anderson-Darling	0.6724 (0.0789)	0.3761 (0.3519)
Cramer-von Mises	0.1219 (0.0564)	0.0535 (0.4301)
Kolmogorov-Smirnov	0.0285 (0.0538)	0.1520 (0.6214)
Pearson chi-square	35.1360 (0.2001)	4.0000 (0.2615)
Shapiro-Francia	0.9973 (0.0878)	0.9183 (0.2293)
Shapiro-Wilk	0.9975 (0.1238)	0.9291 (0.3710)

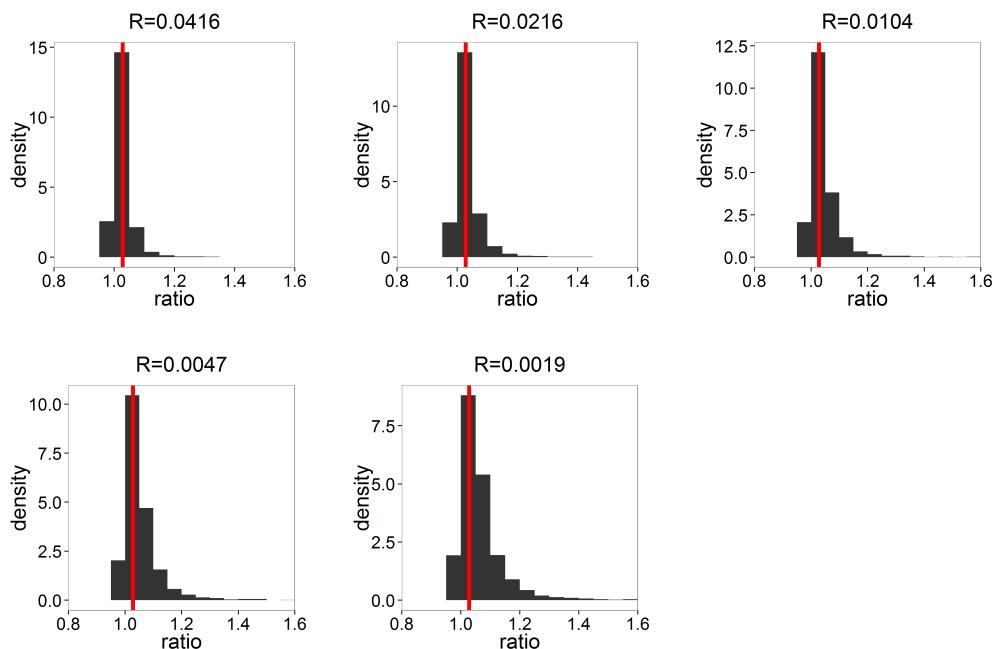
**Figure S1:** Histograms of the ratio of the 95% upper bound to the upper bound of the 90% credible interval of the Bayesian estimator for different values of the true  $R$  and sample sizes  $n_x = 100$  and  $n_y = 12$ . The vertical red line indicates the corresponding ratio in the case study.



**Figure S2:** Histograms of the distances between the 95% upper bound and the true  $R$  value of the QMLE (left column) and Bayesian estimator (right column) for different values of the true  $R$  and sample sizes  $n_x = 100$  and  $n_y = 12$ . The vertical red line indicates the corresponding upper bounds in the case study.

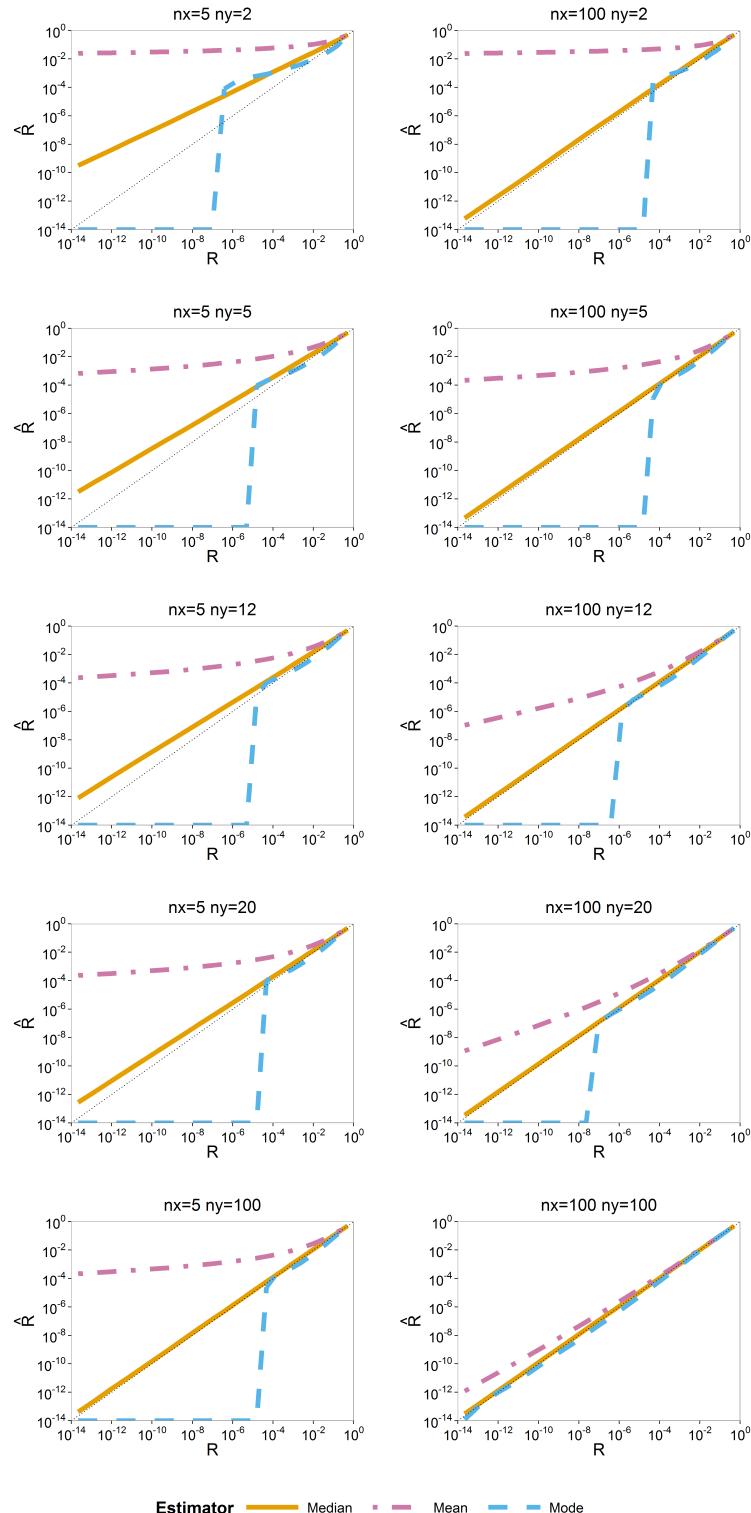


**Figure S3:** Histograms of the ratio of the 95% upper bound of the Bayesian estimator to QMLE for different values of the true  $R$  and sample sizes  $n_x = 100$  and  $n_y = 12$ . The vertical red line indicates the corresponding ratio in the case study.

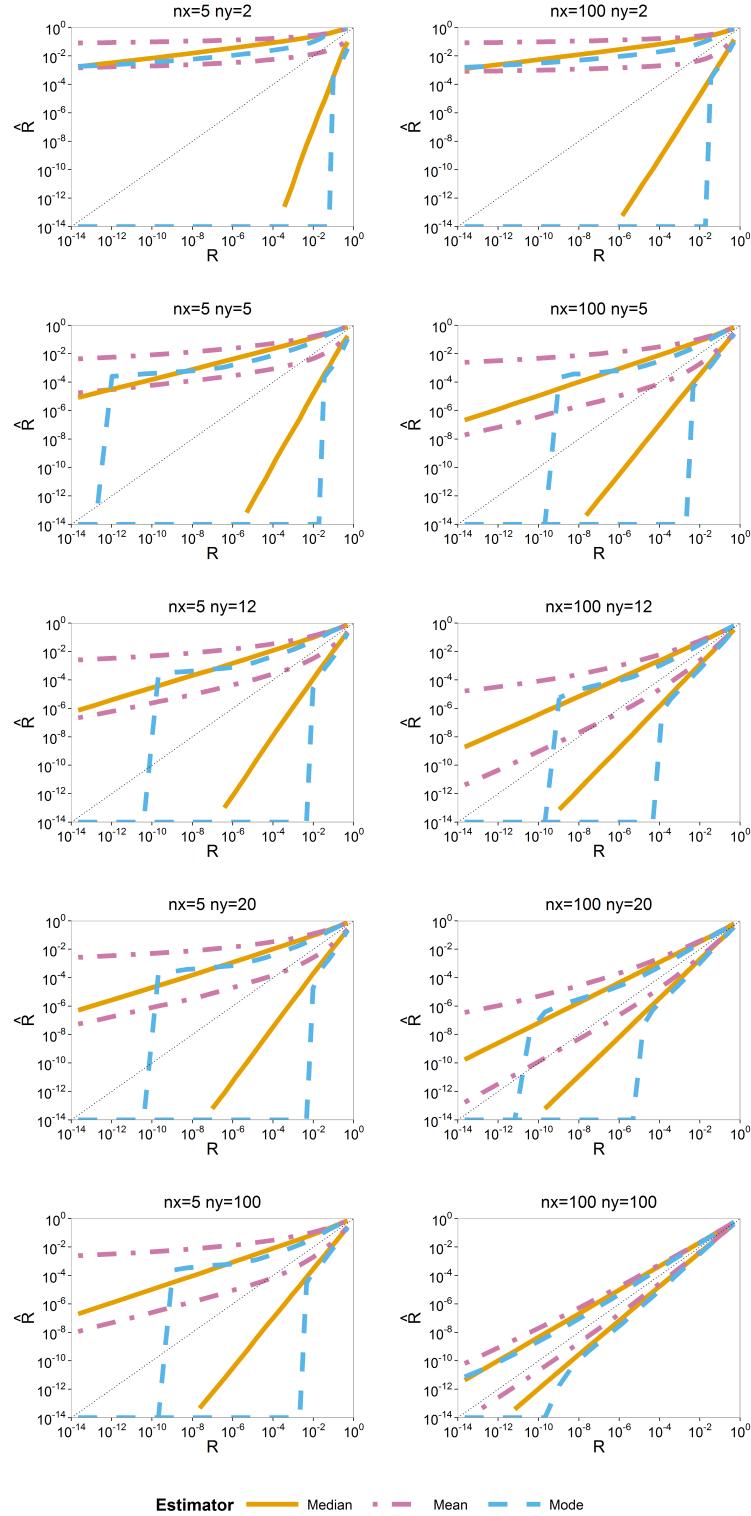


## 2.2 Case: $\sigma_y = \sigma_x$

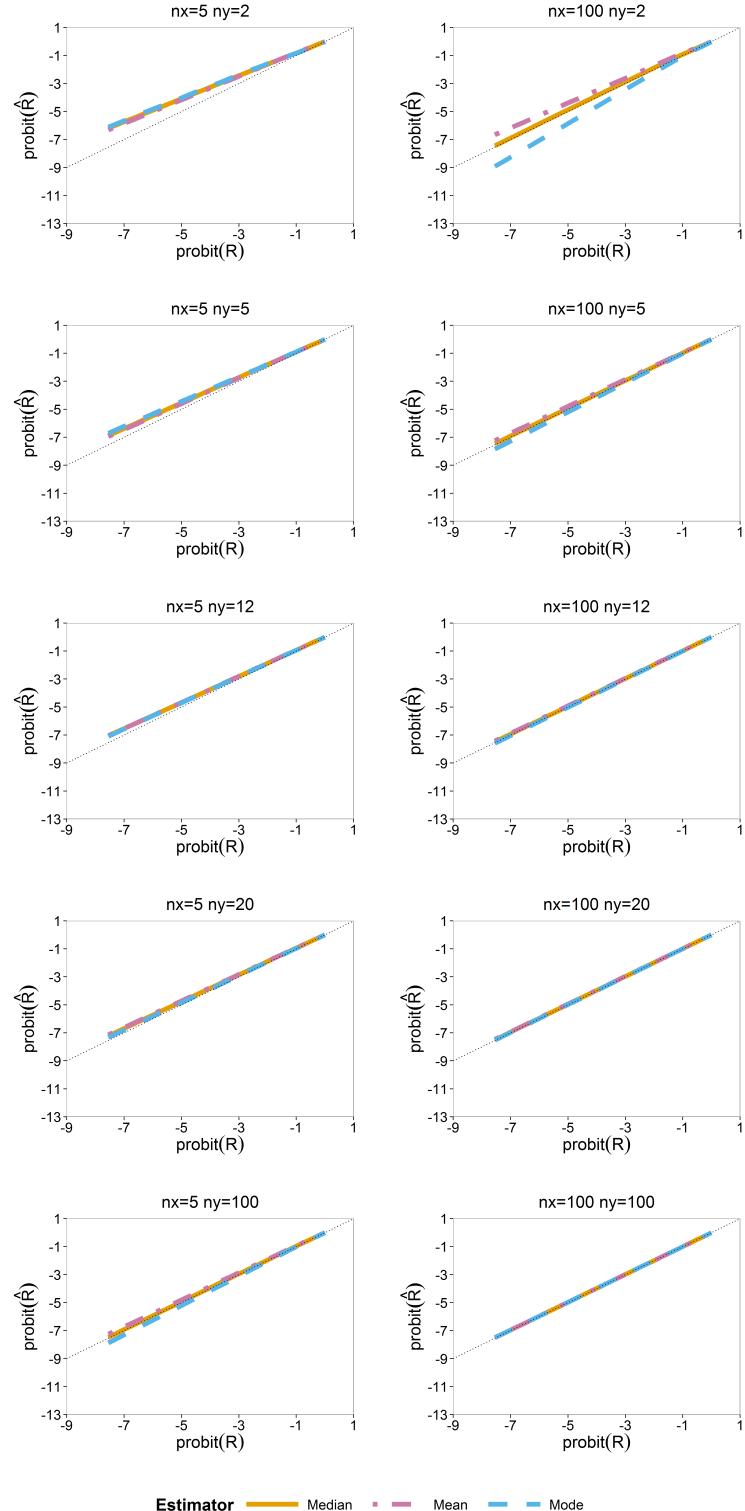
**Figure S4:** The median of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the original scale and plotted on  $\log_{10}$ -scale. When plotted on  $\log_{10}$ -scale, a zero mode becomes  $-\infty$ . The diagonal dotted line represents the values where  $\hat{R} = R$ .



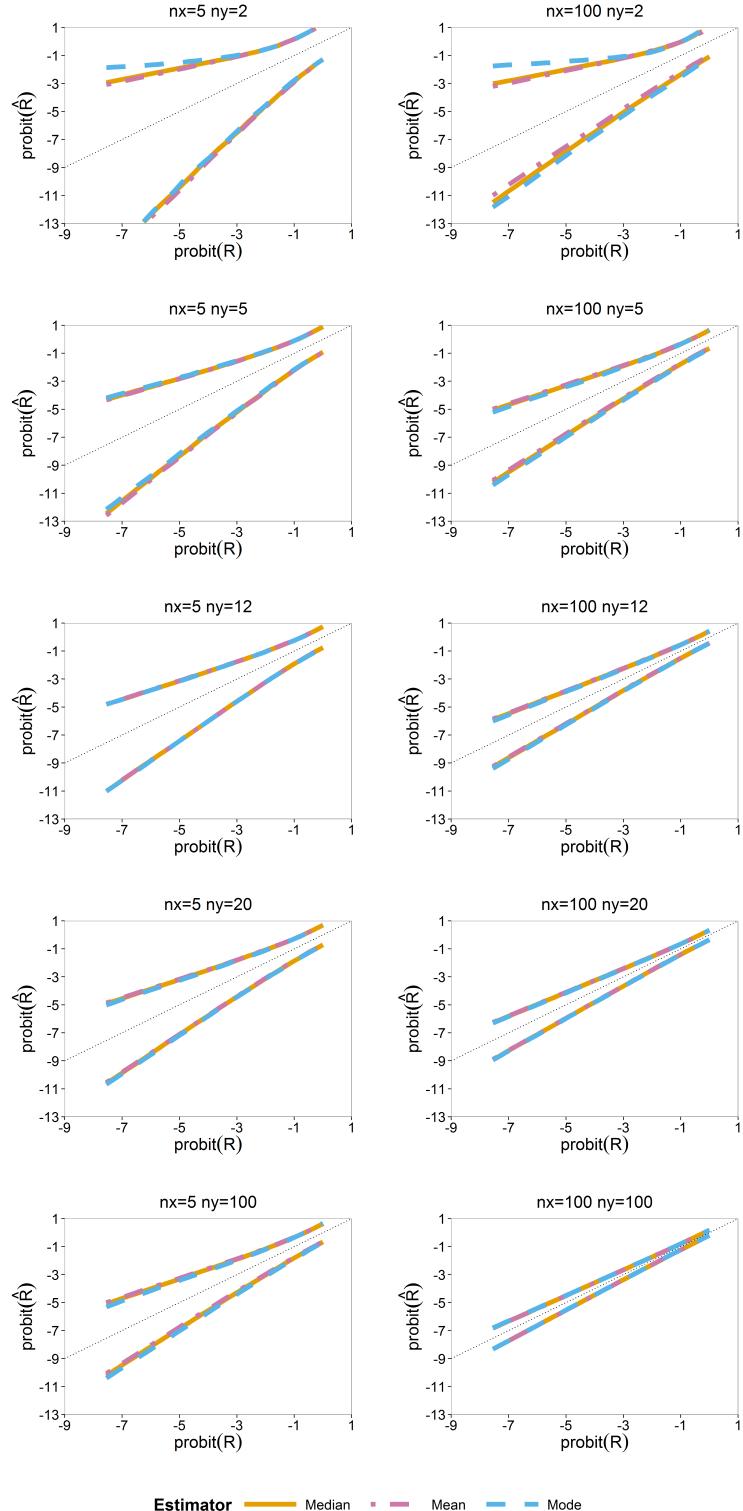
**Figure S5:** The 0.025 and 0.975 quantiles of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the original scale and plotted on  $\log_{10}$ -scale. When plotted on  $\log_{10}$ -scale, a zero mode becomes  $-\infty$ . The diagonal dotted line represents the values where  $\hat{R} = R$ .



**Figure S6:** The median of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the probit scale. The diagonal dotted line represents the values where  $\text{probit}(\hat{R}) = \text{probit}(R)$ .

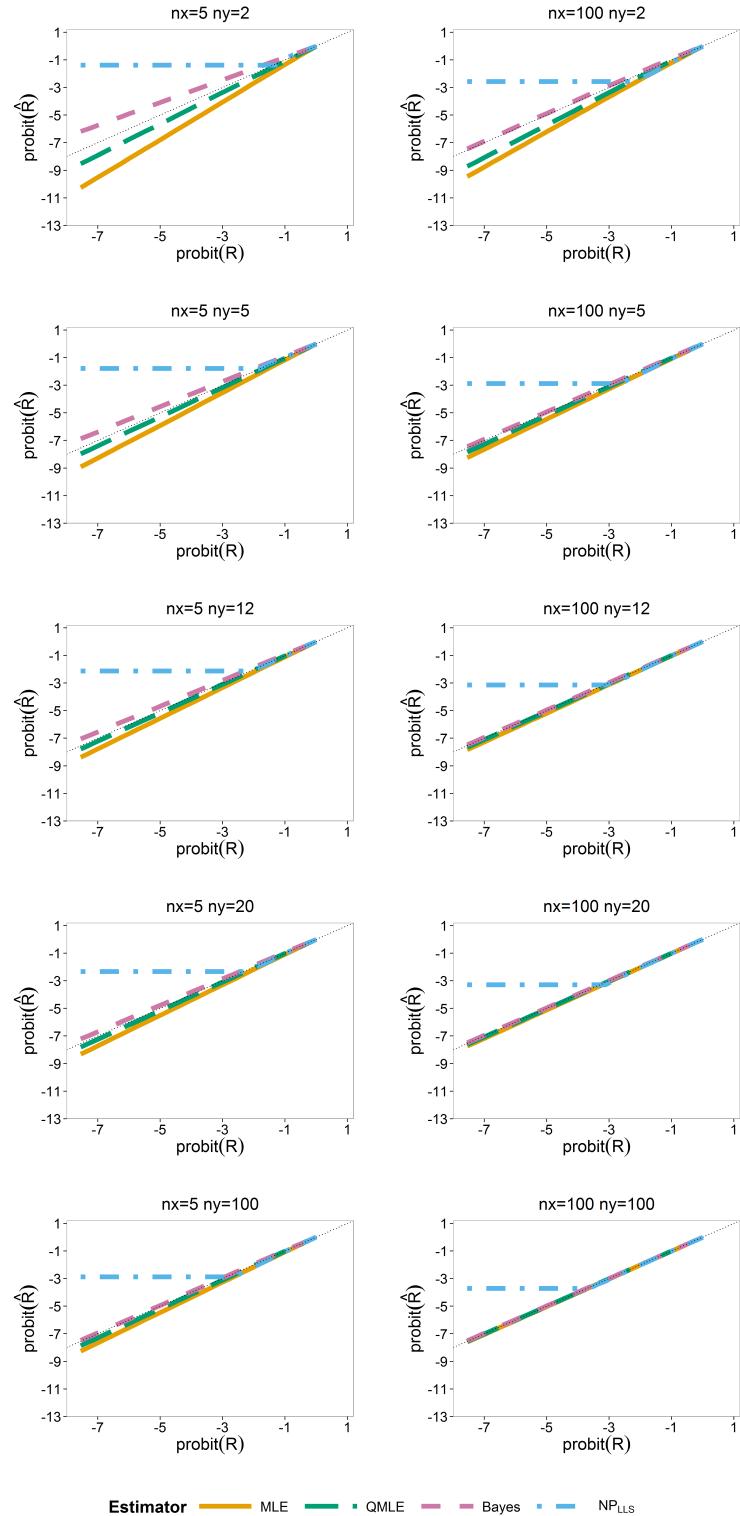


**Figure S7:** The 0.025 and 0.975 quantiles of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the probit scale. The diagonal dotted line represents the values where  $\text{probit}(\hat{R}) = \text{probit}(R)$ .

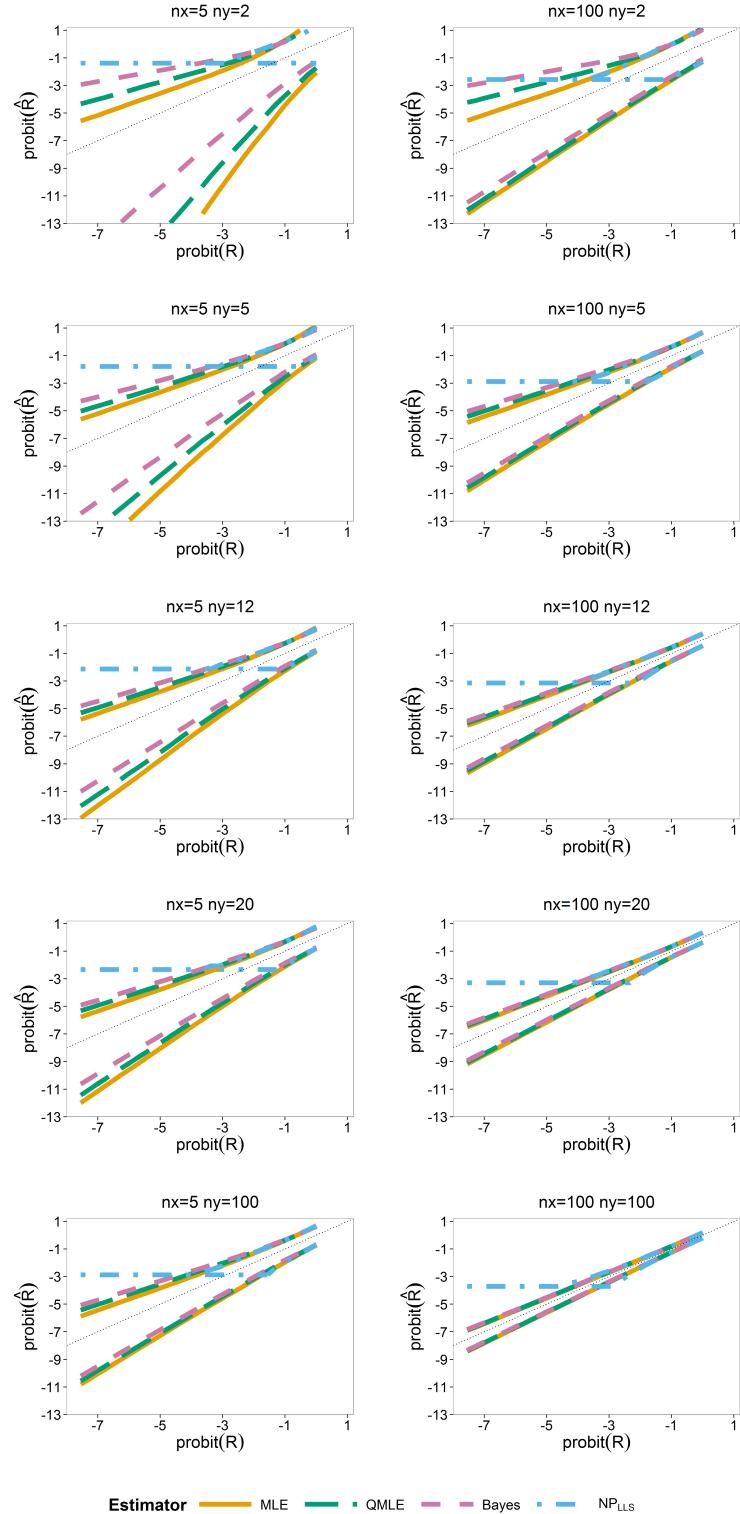


Estimator — Median ■ Mean ■ Mode

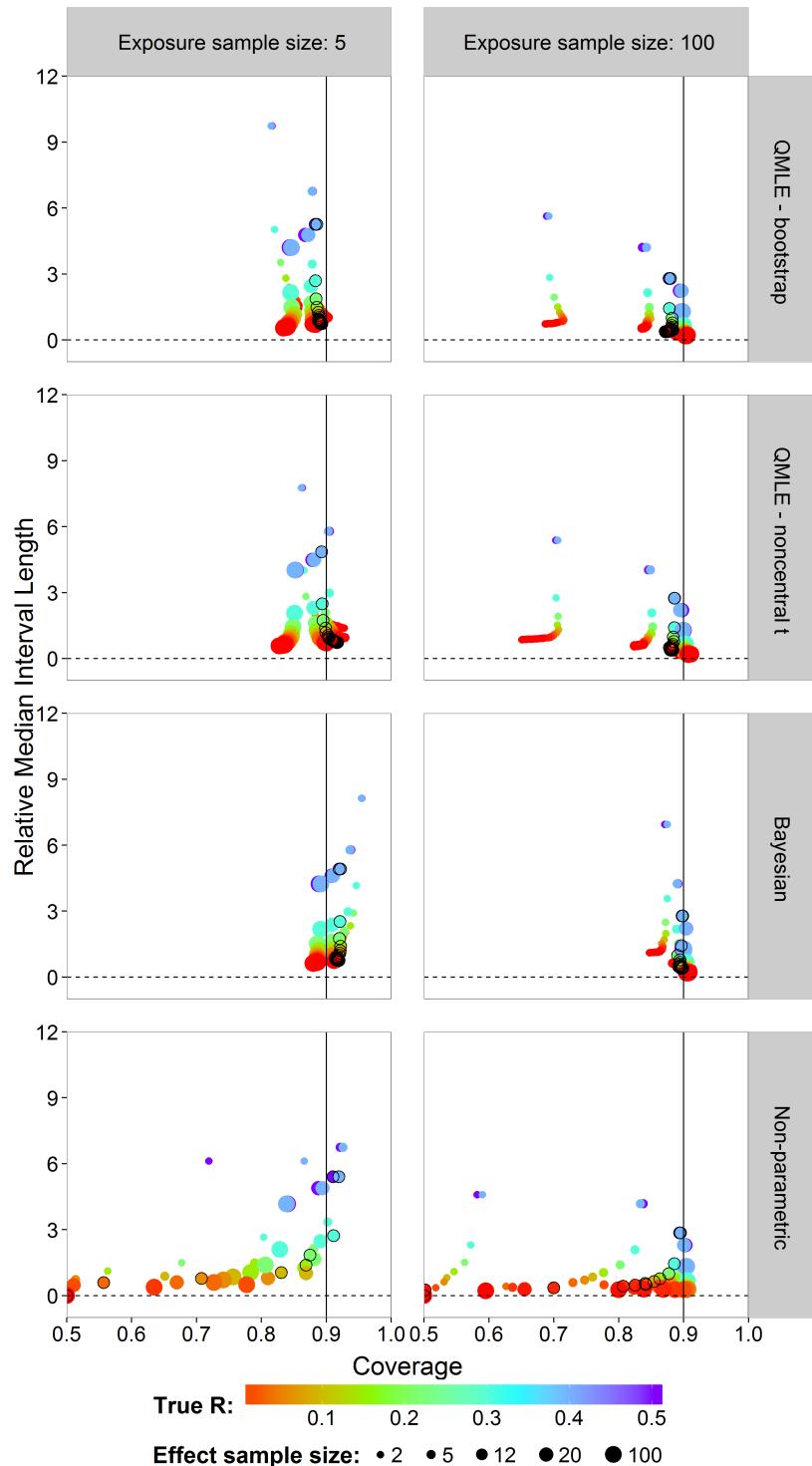
**Figure S8:** The median of the sampling distribution of the four point estimators,  $\hat{R}_{MLE}$ ,  $\hat{R}_{QMLE}$ ,  $\hat{R}_{Bayes}$  and  $\hat{R}_{NP}$  calculated on the probit scale. The diagonal dotted line represents the values where  $probit(\hat{R}) = probit(R)$ . The horizontal grey line gives the lower bound ( $probit\left(\frac{1}{n_x n_y + 2}\right)$ ) of  $probit(\hat{R}_{NP})$ .



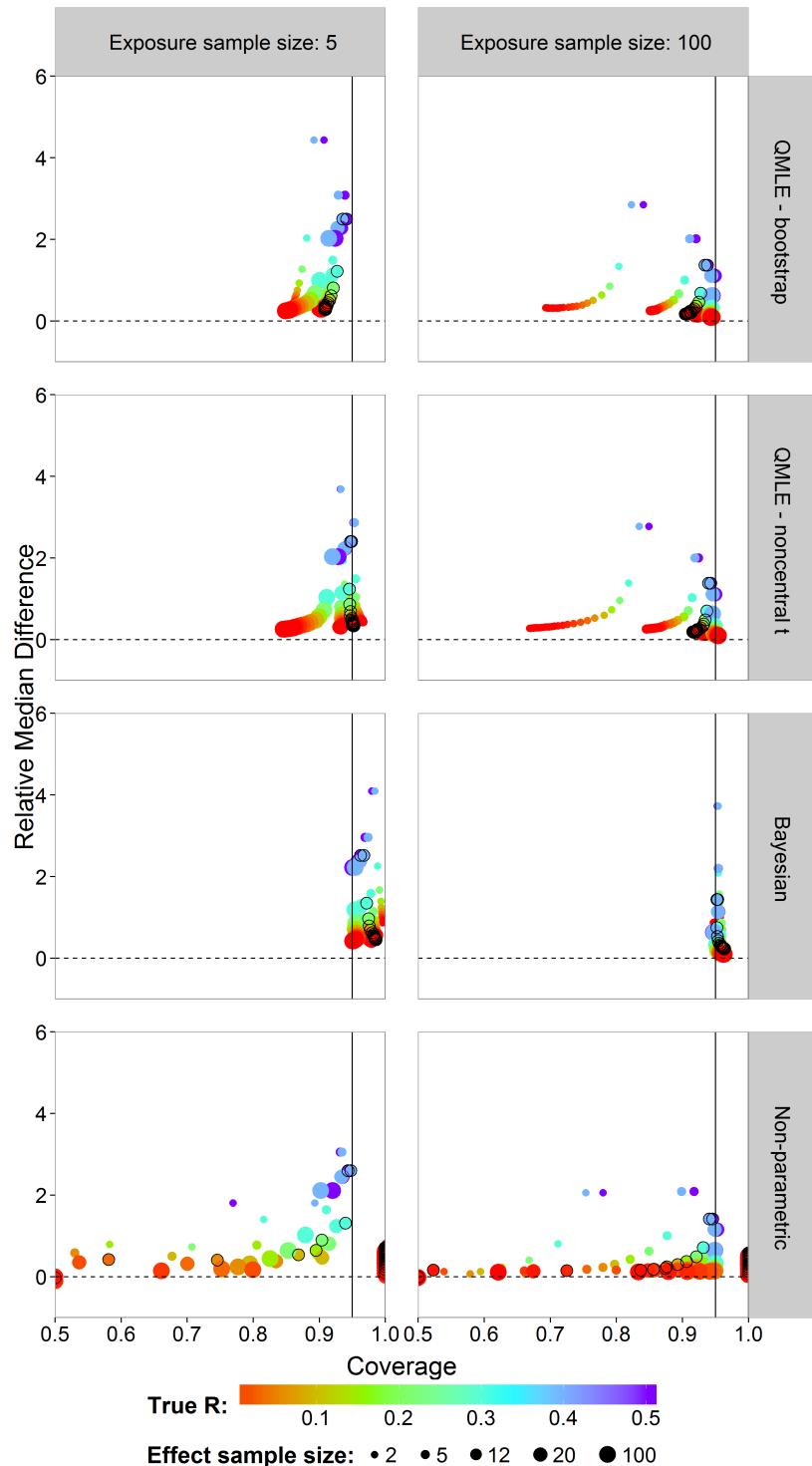
**Figure S9:** The 0.025 and 0.975 quantiles of the sampling distribution of the four point estimators,  $\hat{R}_{MLE}$ ,  $\hat{R}_{QMLE}$ ,  $\hat{R}_{Bayes}$  and  $\hat{R}_{NP}$  calculated on the probit scale. The diagonal dotted line represents the values where  $probit(\hat{R}) = probit(R)$ . The horizontal grey line gives the lower bound ( $probit\left(\frac{1}{nx ny + 2}\right)$ ) of  $probit(\hat{R}_{NP})$ .



**Figure S10:** Scatterplots of the 90% two-sided coverage probabilities against the relative median interval length calculated on the probit scale. The value of the true  $R$  value is illustrated by the color scale. The size of the dots corresponds to the size of the sample size of effect. A vertical reference line is drawn at a coverage probability of 90%. The points corresponding to  $n_y = 12$  are indicated by an open black circle.



**Figure S11:** Scatterplots of the 95% one-sided coverage probabilities against the relative median difference calculated on the probit scale. The value of the true  $R$  value is illustrated by the color scale. The size of the dots corresponds to the size of the sample size of effect. A vertical reference line is drawn at a coverage probability of 95%. The points corresponding to  $n_y = 12$  are indicated by a black circle.



**Table S2:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the MLE for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0139 (1.0115)	-0.0139	1.0116
	0.0966	-1.3014	-1.9791 (1.3210)	-0.6777	1.4847
	0.0021	-2.8632	-4.3374 (2.1167)	-1.4742	2.5795
	4.8e-06	-4.4249	-6.6957 (3.0421)	-2.2707	3.7962
	1.1e-09	-5.9867	-9.0539 (4.0085)	-3.0672	5.0474
	2.2e-14	-7.5484	-11.4122 (4.9920)	-3.8638	6.3126
(5,5)	0.5000	0.0000	-0.0047 (0.5838)	-0.0047	0.5838
	0.0966	-1.3014	-1.6239 (0.7537)	-0.3224	0.8197
	0.0021	-2.8632	-3.5669 (1.1882)	-0.7037	1.3810
	4.8e-06	-4.4249	-5.5100 (1.6967)	-1.0851	2.0140
	1.1e-09	-5.9867	-7.4530 (2.2292)	-1.4664	2.6682
	2.2e-14	-7.5484	-9.3961 (2.7718)	-1.8477	3.3312
(5,12)	0.5000	0.0000	-0.0003 (0.4373)	-0.0003	0.4373
	0.0966	-1.3014	-1.4901 (0.5304)	-0.1887	0.5629
	0.0021	-2.8632	-3.2779 (0.7997)	-0.4147	0.9008
	4.8e-06	-4.4249	-5.0656 (1.1273)	-0.6407	1.2966
	1.1e-09	-5.9867	-6.8534 (1.4747)	-0.8667	1.7105
	2.2e-14	-7.5484	-8.6411 (1.8308)	-1.0927	2.1321
(5,20)	0.5000	0.0000	0.0029 (0.4040)	0.0029	0.4040
	0.0966	-1.3014	-1.4525 (0.4823)	-0.1511	0.5054
	0.0021	-2.8632	-3.1991 (0.7104)	-0.3359	0.7859
	4.8e-06	-4.4249	-4.9457 (0.9914)	-0.5208	1.1198
	1.1e-09	-5.9867	-6.6923 (1.2911)	-0.7056	1.4713
	2.2e-14	-7.5484	-8.4388 (1.5991)	-0.8904	1.8303
(5,100)	0.5000	0.0000	0.0061 (0.3565)	0.0061	0.3565
	0.0966	-1.3014	-1.4208 (0.4181)	-0.1194	0.4348
	0.0021	-2.8632	-3.1331 (0.5940)	-0.2699	0.6525
	4.8e-06	-4.4249	-4.8454 (0.8141)	-0.4205	0.9163
	1.1e-09	-5.9867	-6.5577 (1.0508)	-0.5711	1.1960
	2.2e-14	-7.5484	-8.2701 (1.2952)	-0.7217	1.4827
(100,2)	0.5000	0.0000	-0.0131 (0.6229)	-0.0131	0.6230
	0.0966	-1.3014	-1.6115 (0.6814)	-0.3100	0.7486
	0.0021	-2.8632	-3.5295 (0.8675)	-0.6664	1.0939
	4.8e-06	-4.4249	-5.4476 (1.1203)	-1.0227	1.5169
	1.1e-09	-5.9867	-7.3657 (1.4042)	-1.3790	1.9681
	2.2e-14	-7.5484	-9.2837 (1.7039)	-1.7353	2.4320
(100,5)	0.5000	0.0000	0.0007 (0.3573)	0.0007	0.3573
	0.0966	-1.3014	-1.4213 (0.4144)	-0.1198	0.4313
	0.0021	-2.8632	-3.1277 (0.5879)	-0.2645	0.6447
	4.8e-06	-4.4249	-4.8341 (0.8073)	-0.4091	0.9051
	1.1e-09	-5.9867	-6.5405 (1.0440)	-0.5538	1.1818
	2.2e-14	-7.5484	-8.2469 (1.2885)	-0.6985	1.4656
(100,12)	0.5000	0.0000	0.0014 (0.2256)	0.0014	0.2256
	0.0966	-1.3014	-1.3535 (0.2709)	-0.0520	0.2758
	0.0021	-2.8632	-2.9793 (0.3953)	-0.1162	0.4120
	4.8e-06	-4.4249	-4.6052 (0.5479)	-0.1803	0.5768
	1.1e-09	-5.9867	-6.2311 (0.7107)	-0.2444	0.7515
	2.2e-14	-7.5484	-7.8570 (0.8780)	-0.3086	0.9307
(100,20)	0.5000	0.0000	0.0014 (0.1769)	0.0014	0.1769
	0.0966	-1.3014	-1.3360 (0.2121)	-0.0346	0.2149
	0.0021	-2.8632	-2.9410 (0.3101)	-0.0778	0.3197
	4.8e-06	-4.4249	-4.5459 (0.4304)	-0.1210	0.4471
	1.1e-09	-5.9867	-6.1509 (0.5588)	-0.1642	0.5824
	2.2e-14	-7.5484	-7.7558 (0.6907)	-0.2074	0.7212
(100,100)	0.5000	0.0000	-0.0032 (0.1003)	-0.0032	0.1003
	0.0966	-1.3014	-1.3152 (0.1198)	-0.0137	0.1206
	0.0021	-2.8632	-2.8896 (0.1750)	-0.0264	0.1770
	4.8e-06	-4.4249	-4.4640 (0.2430)	-0.0390	0.2461
	1.1e-09	-5.9867	-6.0384 (0.3156)	-0.0517	0.3198
	2.2e-14	-7.5484	-7.6128 (0.3902)	-0.0644	0.3955

**Table S3:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the QMLE for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0112 (0.8466)	-0.0112	0.8467
	0.0966	-1.3014	-1.6454 (1.1225)	-0.3439	1.1740
	0.0021	-2.8632	-3.6064 (1.8236)	-0.7432	1.9692
	4.8e-06	-4.4249	-5.5674 (2.6331)	-1.1425	2.8703
	1.1e-09	-5.9867	-7.5285 (3.4761)	-1.5418	3.8027
	2.2e-14	-7.5484	-9.4895 (4.3330)	-1.9411	4.7479
(5,5)	0.5000	0.0000	-0.0042 (0.5222)	-0.0042	0.5222
	0.0966	-1.3014	-1.4524 (0.6741)	-0.1510	0.6908
	0.0021	-2.8632	-3.1904 (1.0628)	-0.3272	1.1120
	4.8e-06	-4.4249	-4.9283 (1.5176)	-0.5034	1.5989
	1.1e-09	-5.9867	-6.6662 (1.9938)	-0.6795	2.1065
	2.2e-14	-7.5484	-8.4041 (2.4791)	-0.8557	2.6227
(5,12)	0.5000	0.0000	-0.0003 (0.4068)	-0.0003	0.4068
	0.0966	-1.3014	-1.3845 (0.4970)	-0.0831	0.5039
	0.0021	-2.8632	-3.0456 (0.7554)	-0.1824	0.7771
	4.8e-06	-4.4249	-4.7066 (1.0680)	-0.2817	1.1045
	1.1e-09	-5.9867	-6.3677 (1.3988)	-0.3810	1.4498
	2.2e-14	-7.5484	-8.0288 (1.7376)	-0.4804	1.8028
(5,20)	0.5000	0.0000	0.0026 (0.3804)	0.0026	0.3804
	0.0966	-1.3014	-1.3657 (0.4612)	-0.0643	0.4656
	0.0021	-2.8632	-3.0078 (0.6908)	-0.1446	0.7057
	4.8e-06	-4.4249	-4.6498 (0.9702)	-0.2249	0.9959
	1.1e-09	-5.9867	-6.2918 (1.2668)	-0.3052	1.3030
	2.2e-14	-7.5484	-7.9338 (1.5710)	-0.3854	1.6176
(5,100)	0.5000	0.0000	0.0058 (0.3407)	0.0058	0.3407
	0.0966	-1.3014	-1.3547 (0.4104)	-0.0533	0.4139
	0.0021	-2.8632	-2.9874 (0.6043)	-0.1242	0.6170
	4.8e-06	-4.4249	-4.6201 (0.8415)	-0.1952	0.8638
	1.1e-09	-5.9867	-6.2528 (1.0941)	-0.2661	1.1260
	2.2e-14	-7.5484	-7.8854 (1.3535)	-0.3370	1.3948
(100,2)	0.5000	0.0000	-0.0121 (0.5763)	-0.0121	0.5765
	0.0966	-1.3014	-1.4736 (0.6726)	-0.1721	0.6943
	0.0021	-2.8632	-3.2273 (0.9523)	-0.3641	1.0195
	4.8e-06	-4.4249	-4.9810 (1.3036)	-0.5561	1.4172
	1.1e-09	-5.9867	-6.7347 (1.6823)	-0.7480	1.8411
	2.2e-14	-7.5484	-8.4884 (2.0734)	-0.9400	2.2766
(100,5)	0.5000	0.0000	0.0007 (0.3414)	0.0007	0.3414
	0.0966	-1.3014	-1.3549 (0.4068)	-0.0534	0.4103
	0.0021	-2.8632	-2.9816 (0.5989)	-0.1184	0.6105
	4.8e-06	-4.4249	-4.6083 (0.8358)	-0.1834	0.8557
	1.1e-09	-5.9867	-6.2351 (1.0887)	-0.2484	1.1166
	2.2e-14	-7.5484	-7.8618 (1.3485)	-0.3134	1.3844
(100,12)	0.5000	0.0000	0.0014 (0.2205)	0.0014	0.2205
	0.0966	-1.3014	-1.3229 (0.2672)	-0.0215	0.2681
	0.0021	-2.8632	-2.9121 (0.3942)	-0.0489	0.3973
	4.8e-06	-4.4249	-4.5012 (0.5489)	-0.0763	0.5542
	1.1e-09	-5.9867	-6.0904 (0.7135)	-0.1037	0.7210
	2.2e-14	-7.5484	-7.6796 (0.8824)	-0.1312	0.8921
(100,20)	0.5000	0.0000	0.0014 (0.1743)	0.0014	0.1743
	0.0966	-1.3014	-1.3162 (0.2099)	-0.0148	0.2104
	0.0021	-2.8632	-2.8974 (0.3084)	-0.0342	0.3102
	4.8e-06	-4.4249	-4.4786 (0.4289)	-0.0537	0.4322
	1.1e-09	-5.9867	-6.0598 (0.5573)	-0.0731	0.5621
	2.2e-14	-7.5484	-7.6409 (0.6892)	-0.0925	0.6954
(100,100)	0.5000	0.0000	-0.0032 (0.0998)	-0.0032	0.0998
	0.0966	-1.3014	-1.3086 (0.1192)	-0.0071	0.1194
	0.0021	-2.8632	-2.8751 (0.1741)	-0.0119	0.1745
	4.8e-06	-4.4249	-4.4416 (0.2417)	-0.0167	0.2423
	1.1e-09	-5.9867	-6.0081 (0.3140)	-0.0214	0.3147
	2.2e-14	-7.5484	-7.5746 (0.3883)	-0.0262	0.3891

**Table S4:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the Bayes estimator for all sample size scenarios and selected  $R$  values.

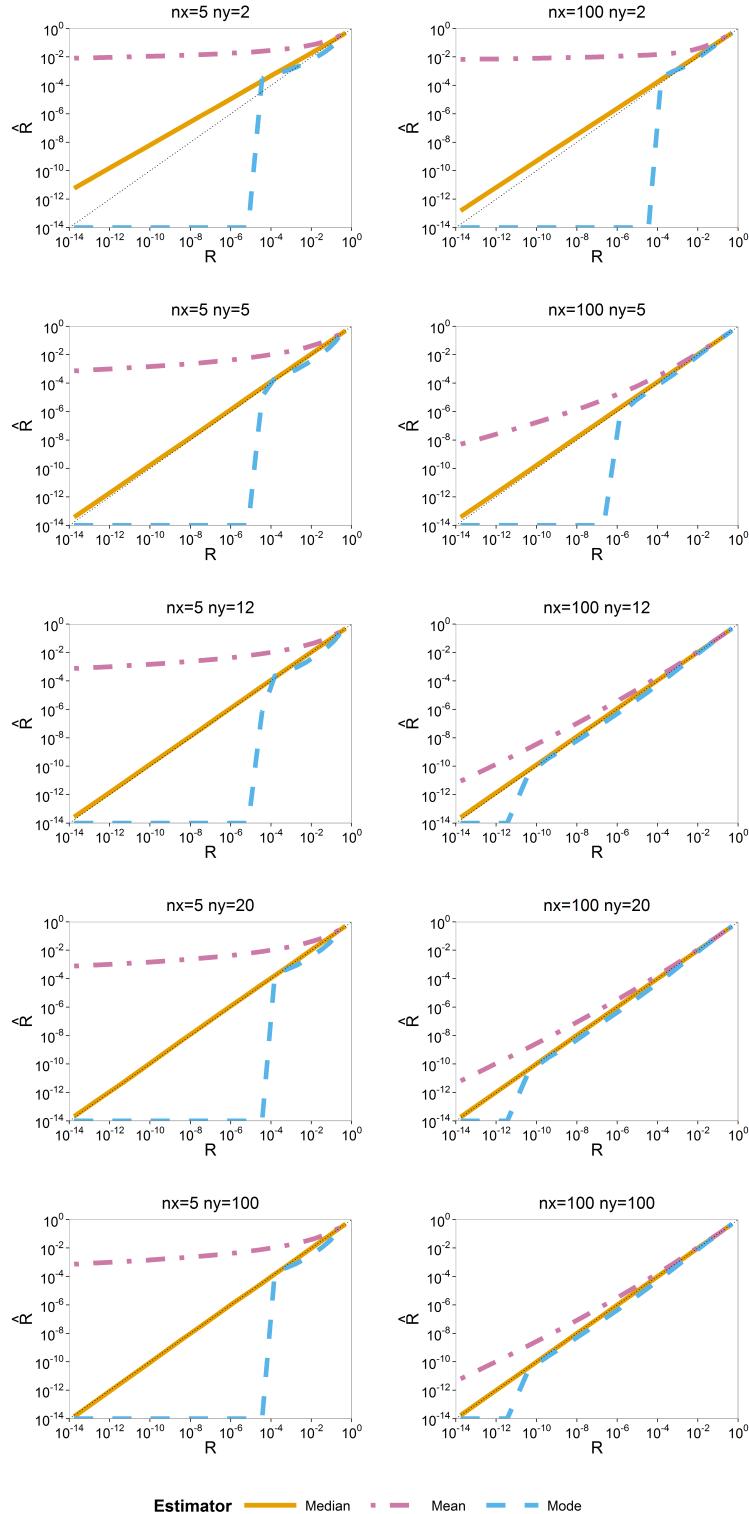
$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0080 (0.6408)	-0.0080	0.6409
	0.0966	-1.3014	-1.2282 (0.8496)	0.0733	0.8528
	0.0021	-2.8632	-2.6603 (1.4094)	0.2029	1.4240
	4.8e-06	-4.4249	-4.0899 (2.0566)	0.3350	2.0837
	1.1e-09	-5.9867	-5.5223 (2.7272)	0.4643	2.7665
	2.2e-14	-7.5484	-6.9561 (3.4069)	0.5923	3.4580
(5,5)	0.5000	0.0000	-0.0037 (0.4523)	-0.0037	0.4523
	0.0966	-1.3014	-1.2547 (0.5796)	0.0468	0.5815
	0.0021	-2.8632	-2.7466 (0.9129)	0.1166	0.9203
	4.8e-06	-4.4249	-4.2376 (1.3041)	0.1873	1.3175
	1.1e-09	-5.9867	-5.7290 (1.7140)	0.2577	1.7332
	2.2e-14	-7.5484	-7.2208 (2.1317)	0.3276	2.1567
(5,12)	0.5000	0.0000	-0.0003 (0.3716)	-0.0003	0.3716
	0.0966	-1.3014	-1.2611 (0.4539)	0.0404	0.4557
	0.0021	-2.8632	-2.7691 (0.6922)	0.0941	0.6986
	4.8e-06	-4.4249	-4.2768 (0.9804)	0.1481	0.9915
	1.1e-09	-5.9867	-5.7848 (1.2852)	0.2018	1.3010
	2.2e-14	-7.5484	-7.2930 (1.5972)	0.2554	1.6175
(5,20)	0.5000	0.0000	0.0003 (0.3534)	0.0023	0.3535
	0.0966	-1.3014	-1.2668 (0.4310)	0.0347	0.4324
	0.0021	-2.8632	-2.7862 (0.6504)	0.0770	0.6549
	4.8e-06	-4.4249	-4.3054 (0.9161)	0.1196	0.9238
	1.1e-09	-5.9867	-5.8245 (1.1976)	0.1621	1.2086
	2.2e-14	-7.5484	-7.3440 (1.4860)	0.2044	1.5000
(5,100)	0.5000	0.0000	0.0056 (0.3249)	0.0056	0.3250
	0.0966	-1.3014	-1.2898 (0.3977)	0.0117	0.3979
	0.0021	-2.8632	-2.8445 (0.5977)	0.0187	0.5980
	4.8e-06	-4.4249	-4.3995 (0.8393)	0.0254	0.8397
	1.1e-09	-5.9867	-5.9547 (1.0953)	0.0320	1.0957
	2.2e-14	-7.5484	-7.5098 (1.3574)	0.0386	1.3579
(100,2)	0.5000	0.0000	-0.0105 (0.5136)	-0.0105	0.5137
	0.0966	-1.3014	-1.2868 (0.6395)	0.0147	0.6397
	0.0021	-2.8632	-2.7992 (1.0052)	0.0640	1.0072
	4.8e-06	-4.4249	-4.3101 (1.4448)	0.1148	1.4493
	1.1e-09	-5.9867	-5.8226 (1.9050)	0.1641	1.9121
	2.2e-14	-7.5484	-7.3361 (2.3732)	0.2123	2.3827
(100,5)	0.5000	0.0000	0.0008 (0.3255)	0.0008	0.3255
	0.0966	-1.3014	-1.2898 (0.3942)	0.0116	0.3944
	0.0021	-2.8632	-2.8387 (0.5928)	0.0244	0.5933
	4.8e-06	-4.4249	-4.3879 (0.8344)	0.0370	0.8352
	1.1e-09	-5.9867	-5.9371 (1.0909)	0.0496	1.0921
	2.2e-14	-7.5484	-7.4863 (1.3538)	0.0621	1.3552
(100,12)	0.5000	0.0000	0.0014 (0.2156)	0.0014	0.2156
	0.0966	-1.3014	-1.2928 (0.2620)	0.0086	0.2621
	0.0021	-2.8632	-2.8457 (0.3876)	0.0175	0.3880
	4.8e-06	-4.4249	-4.3987 (0.5402)	0.0263	0.5408
	1.1e-09	-5.9867	-5.9516 (0.7025)	0.0351	0.7033
	2.2e-14	-7.5484	-7.5045 (0.8689)	0.0439	0.8700
(100,20)	0.5000	0.0000	0.0014 (0.1716)	0.0014	0.1716
	0.0966	-1.3014	-1.2954 (0.2067)	0.0061	0.2068
	0.0021	-2.8632	-2.8512 (0.3038)	0.0120	0.3041
	4.8e-06	-4.4249	-4.4070 (0.4226)	0.0179	0.4229
	1.1e-09	-5.9867	-5.9628 (0.5490)	0.0239	0.5495
	2.2e-14	-7.5484	-7.5185 (0.6789)	0.0299	0.6796
(100,100)	0.5000	0.0000	-0.0031 (0.0991)	-0.0031	0.0992
	0.0966	-1.3014	-1.3002 (0.1184)	0.0012	0.1184
	0.0021	-2.8632	-2.8563 (0.1729)	0.0069	0.1731
	4.8e-06	-4.4249	-4.4124 (0.2401)	0.0125	0.2405
	1.1e-09	-5.9867	-5.9684 (0.3119)	0.0183	0.3124
	2.2e-14	-7.5484	-7.5245 (0.3857)	0.0239	0.3865

**Table S5:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the non-parametric estimator for all sample size scenarios and selected  $R$  values.

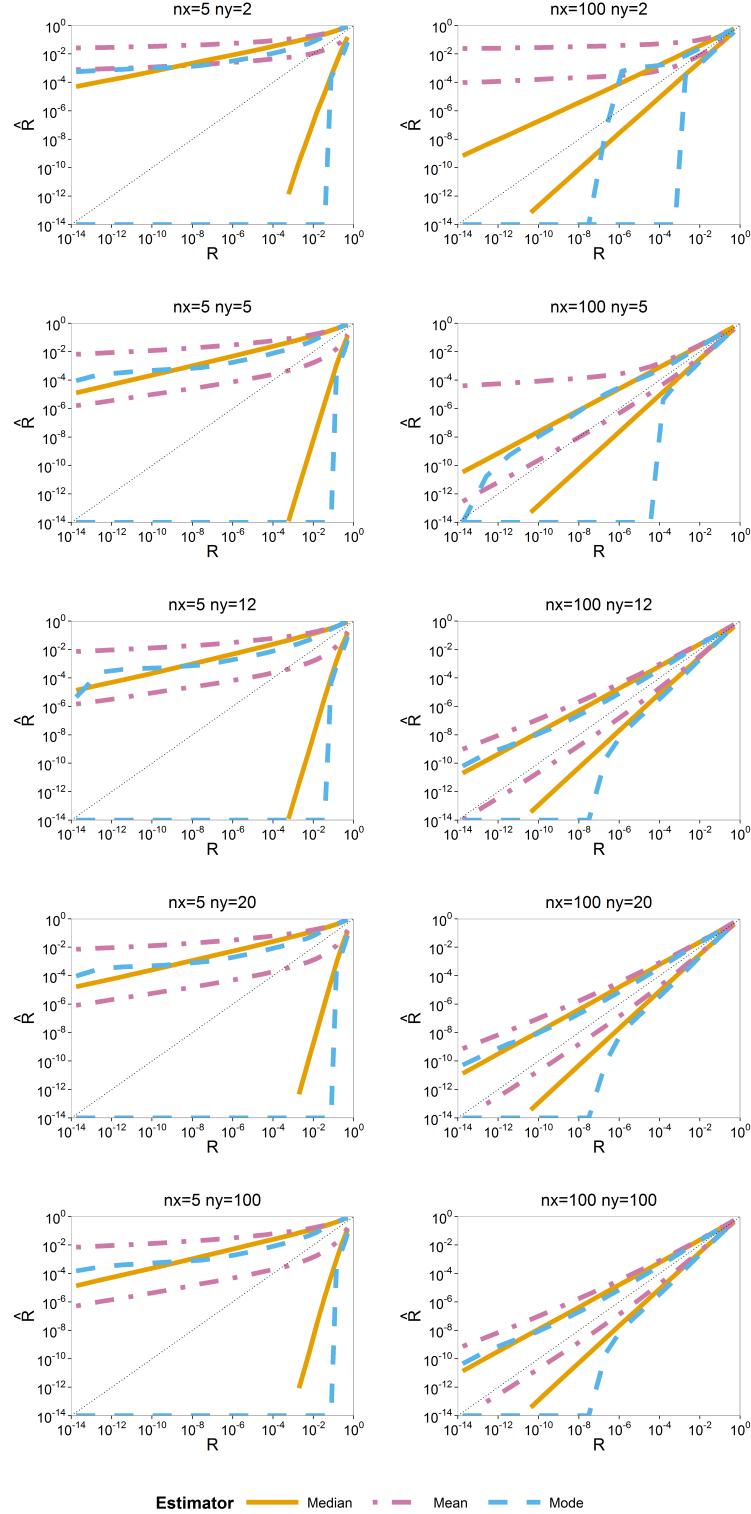
$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0084 (0.6513)	-0.0084	0.6513
	0.0966	-1.3014	-1.0585 (0.4276)	0.2429	0.4918
	0.0021	-2.8632	-1.3759 (0.0594)	1.4873	1.4885
	4.8e-06	-4.4249	-1.3830 (0.0000)	3.0419	3.0419
	1.1e-09	-5.9867	-1.3830 (0.0000)	4.6037	4.6037
	2.2e-14	-7.5484	-1.3830 (0.0000)	6.1654	6.1654
(5,5)	0.5000	0.0000	-0.0037 (0.5129)	-0.0037	0.5130
	0.0966	-1.3014	-1.2666 (0.4514)	0.0349	0.4527
	0.0021	-2.8632	-1.7699 (0.0811)	1.0933	1.0963
	4.8e-06	-4.4249	-1.7862 (0.0000)	2.6388	2.6388
	1.1e-09	-5.9867	-1.7862 (0.0000)	4.2005	4.2005
	2.2e-14	-7.5484	-1.7862 (0.0000)	5.7622	5.7622
(5,12)	0.5000	0.0000	0.0006 (0.4164)	0.0006	0.4164
	0.0966	-1.3014	-1.3553 (0.4597)	-0.0538	0.4628
	0.0021	-2.8632	-2.1091 (0.1136)	0.7540	0.7625
	4.8e-06	-4.4249	-2.1411 (0.0044)	2.2838	2.2838
	1.1e-09	-5.9867	-2.1412 (0.0000)	3.8455	3.8455
	2.2e-14	-7.5484	-2.1412 (0.0000)	5.4072	5.4072
(5,20)	0.5000	0.0000	0.0044 (0.3933)	0.0044	0.3933
	0.0966	-1.3014	-1.3763 (0.4579)	-0.0748	0.4639
	0.0021	-2.8632	-2.2877 (0.1397)	0.5755	0.5922
	4.8e-06	-4.4249	-2.3335 (0.0091)	2.0915	2.0915
	1.1e-09	-5.9867	-2.3338 (0.0000)	3.6529	3.6529
	2.2e-14	-7.5484	-2.3338 (0.0000)	5.2146	5.2146
(5,100)	0.5000	0.0000	0.0062 (0.3500)	0.0062	0.3501
	0.0966	-1.3014	-1.4054 (0.4421)	-0.1040	0.4541
	0.0021	-2.8632	-2.7503 (0.2376)	0.1129	0.2631
	4.8e-06	-4.4249	-2.8787 (0.0204)	1.5462	1.5463
	1.1e-09	-5.9867	-2.8794 (0.0000)	3.1072	3.1072
	2.2e-14	-7.5484	-2.8794 (0.0000)	4.6690	4.6690
(100,2)	0.5000	0.0000	-0.0139 (0.6039)	-0.0139	0.6040
	0.0966	-1.3014	-1.5296 (0.6232)	-0.2281	0.6637
	0.0021	-2.8632	-2.5186 (0.1802)	0.3446	0.3888
	4.8e-06	-4.4249	-2.5789 (0.0098)	1.8460	1.8461
	1.1e-09	-5.9867	-2.5793 (0.0000)	3.4074	3.4074
	2.2e-14	-7.5484	-2.5793 (0.0000)	4.9691	4.9691
(100,5)	0.5000	0.0000	0.0009 (0.3515)	0.0009	0.3515
	0.0966	-1.3014	-1.4019 (0.4353)	-0.1004	0.4467
	0.0021	-2.8632	-2.7463 (0.2379)	0.1169	0.2651
	4.8e-06	-4.4249	-2.8790 (0.0104)	1.5459	1.5460
	1.1e-09	-5.9867	-2.8794 (0.0000)	3.1072	3.1072
	2.2e-14	-7.5484	-2.8794 (0.0000)	4.6690	4.6690
(100,12)	0.5000	0.0000	0.0025 (0.2257)	0.0025	0.2257
	0.0966	-1.3014	-1.3456 (0.2849)	-0.0442	0.2883
	0.0021	-2.8632	-2.8960 (0.2942)	-0.0328	0.2961
	4.8e-06	-4.4249	-3.1435 (0.0156)	1.2814	1.2815
	1.1e-09	-5.9867	-3.1445 (0.0000)	2.8422	2.8422
	2.2e-14	-7.5484	-3.1445 (0.0000)	4.4039	4.4039
(100,20)	0.5000	0.0000	0.0021 (0.1791)	0.0021	0.1791
	0.0966	-1.3014	-1.3303 (0.2215)	-0.0289	0.2234
	0.0021	-2.8632	-2.9462 (0.3093)	-0.0830	0.3202
	4.8e-06	-4.4249	-3.2889 (0.0215)	1.1360	1.1362
	1.1e-09	-5.9867	-3.2908 (0.0000)	2.6959	2.6959
	2.2e-14	-7.5484	-3.2908 (0.0000)	4.2576	4.2576
(100,100)	0.5000	0.0000	-0.0027 (0.1023)	-0.0027	0.1023
	0.0966	-1.3014	-1.3136 (0.1233)	-0.0122	0.1239
	0.0021	-2.8632	-2.9603 (0.3001)	-0.0971	0.3155
	4.8e-06	-4.4249	-3.7115 (0.0420)	0.7134	0.7147
	1.1e-09	-5.9867	-3.7191 (0.0000)	2.2676	2.2676
	2.2e-14	-7.5484	-3.7191 (0.0000)	3.8293	3.8293

### 2.3 Case: $\sigma_y = 0.2\sigma_x$

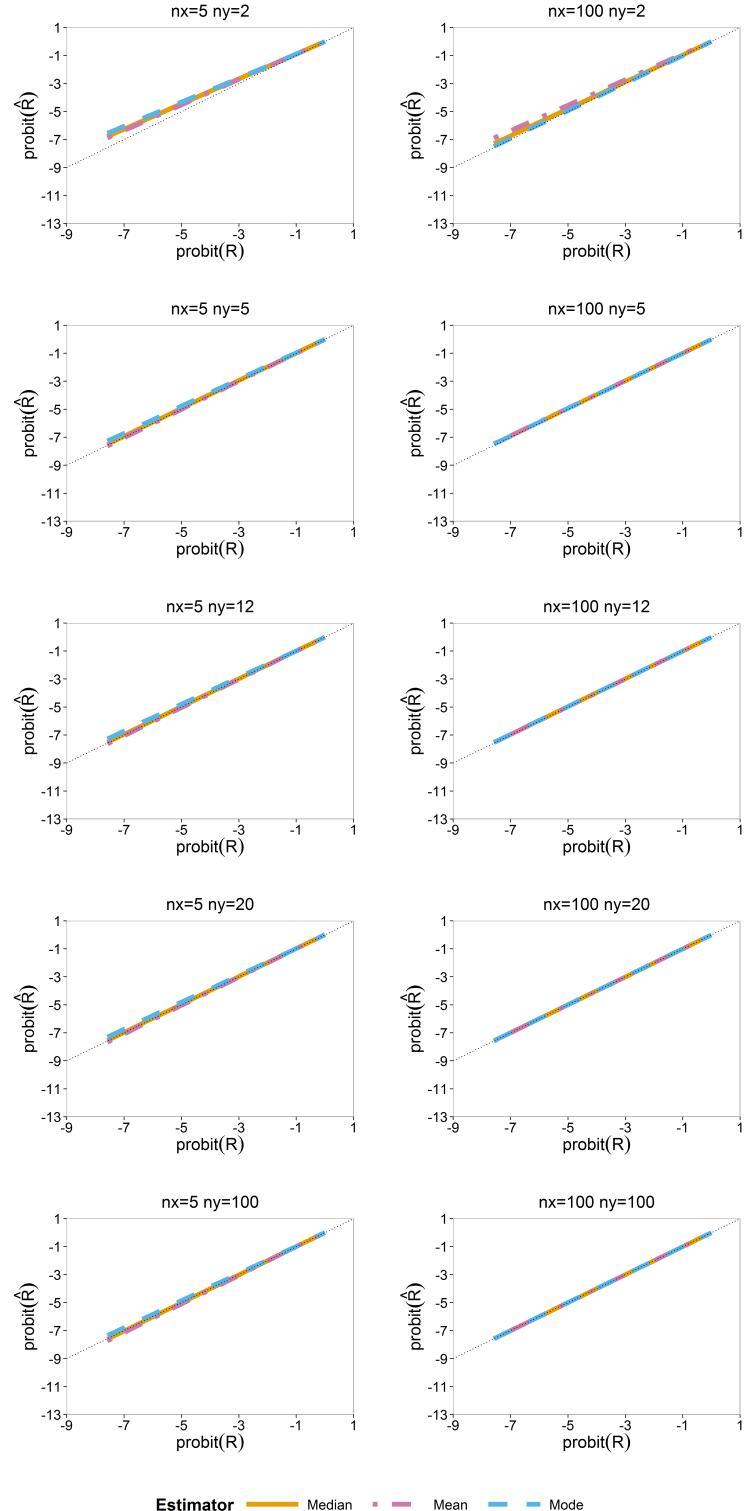
**Figure S12:** The median of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the original scale and plotted on  $\log_{10}$ -scale. When plotted on  $\log_{10}$ -scale, a zero mode becomes  $-\infty$ . The diagonal dotted line represents the values where  $\hat{R} = R$ .



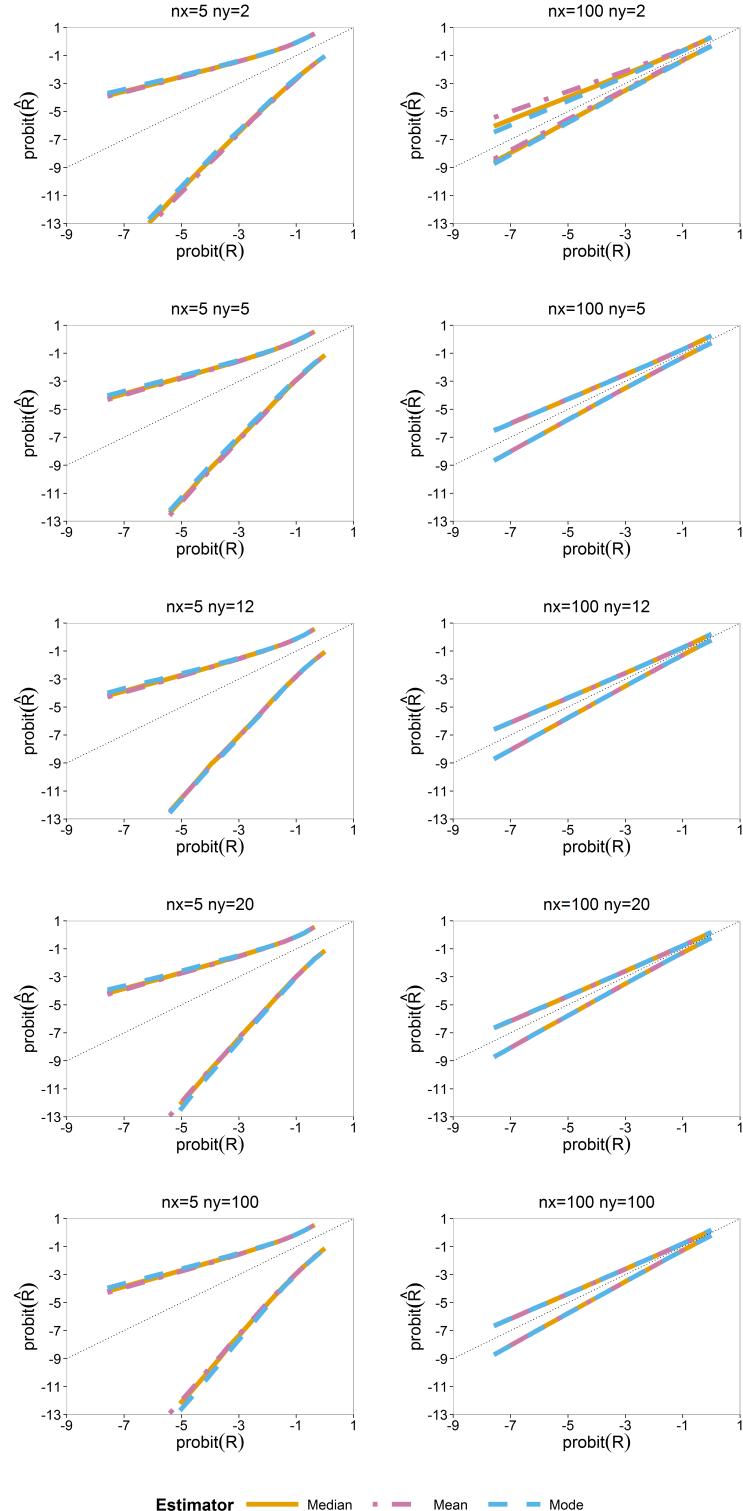
**Figure S13:** The 0.025 and 0.975 quantiles of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the original scale and plotted on  $\log_{10}$ -scale. When plotted on  $\log_{10}$ -scale, a zero mode becomes  $-\infty$ . The diagonal dotted line represents the values where  $\hat{R} = R$ .



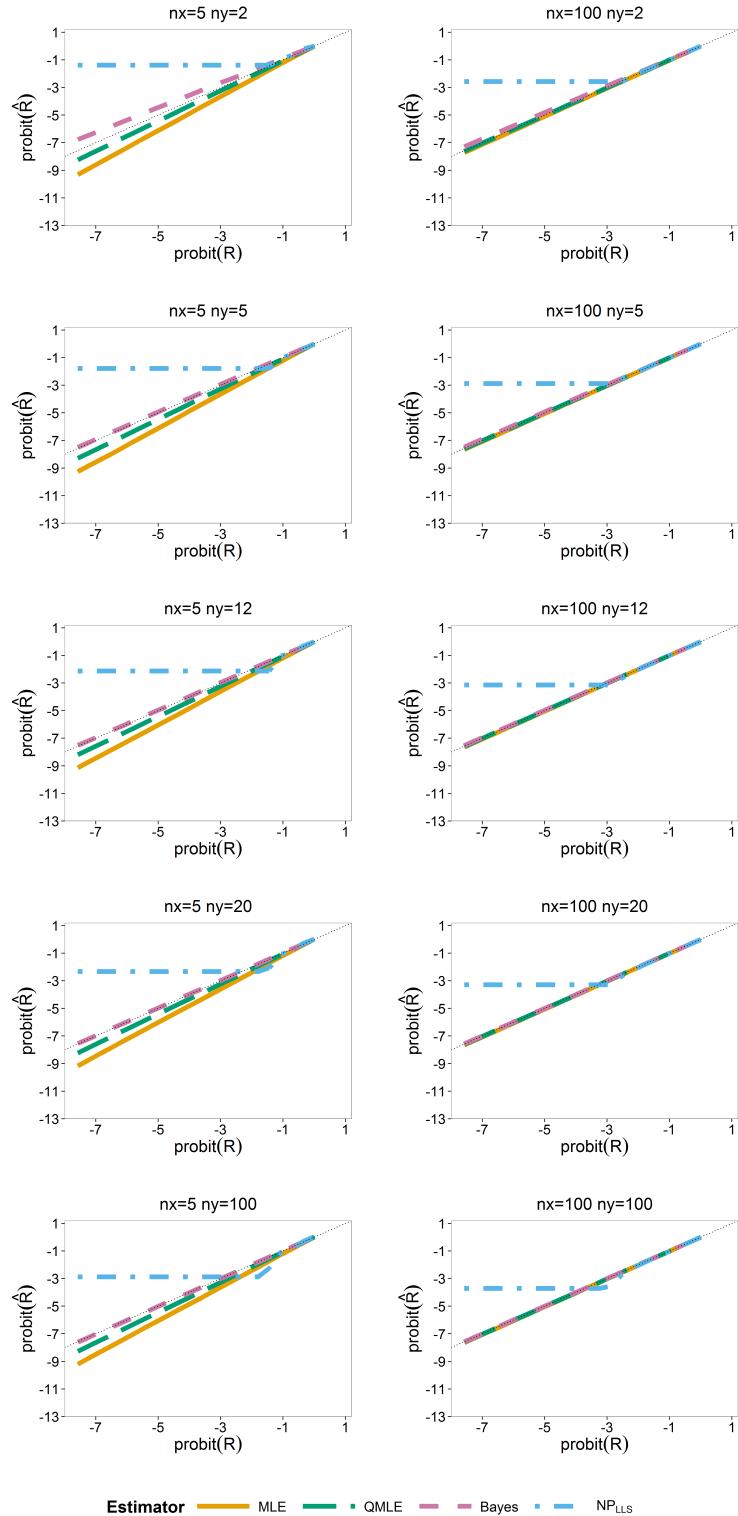
**Figure S14:** The median of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the probit scale. The diagonal dotted line represents the values where  $\text{probit}(\hat{R}) = \text{probit}(R)$ .



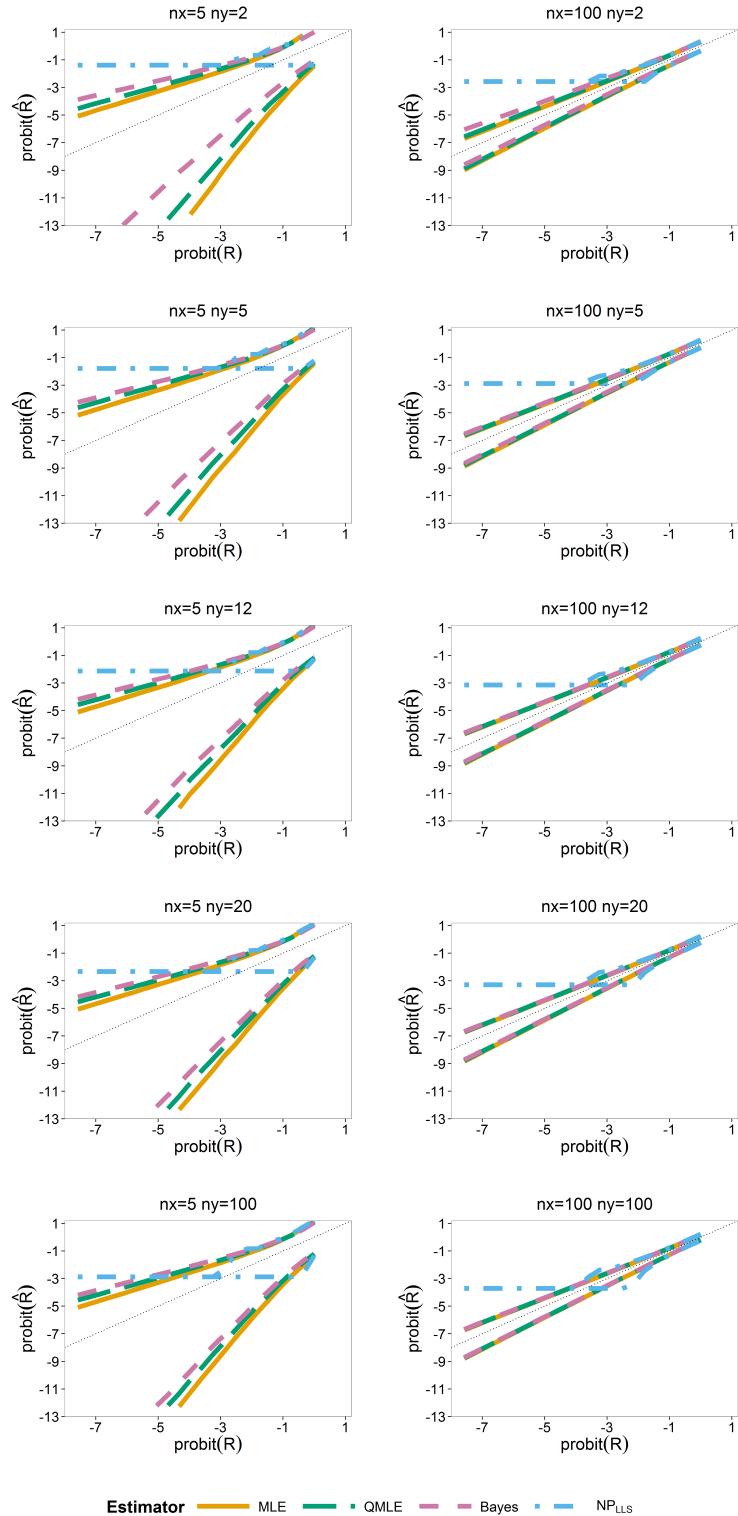
**Figure S15:** The 0.025 and 0.975 quantiles of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the probit scale. The diagonal dotted line represents the values where  $\text{probit}(\hat{R}) = \text{probit}(R)$ .



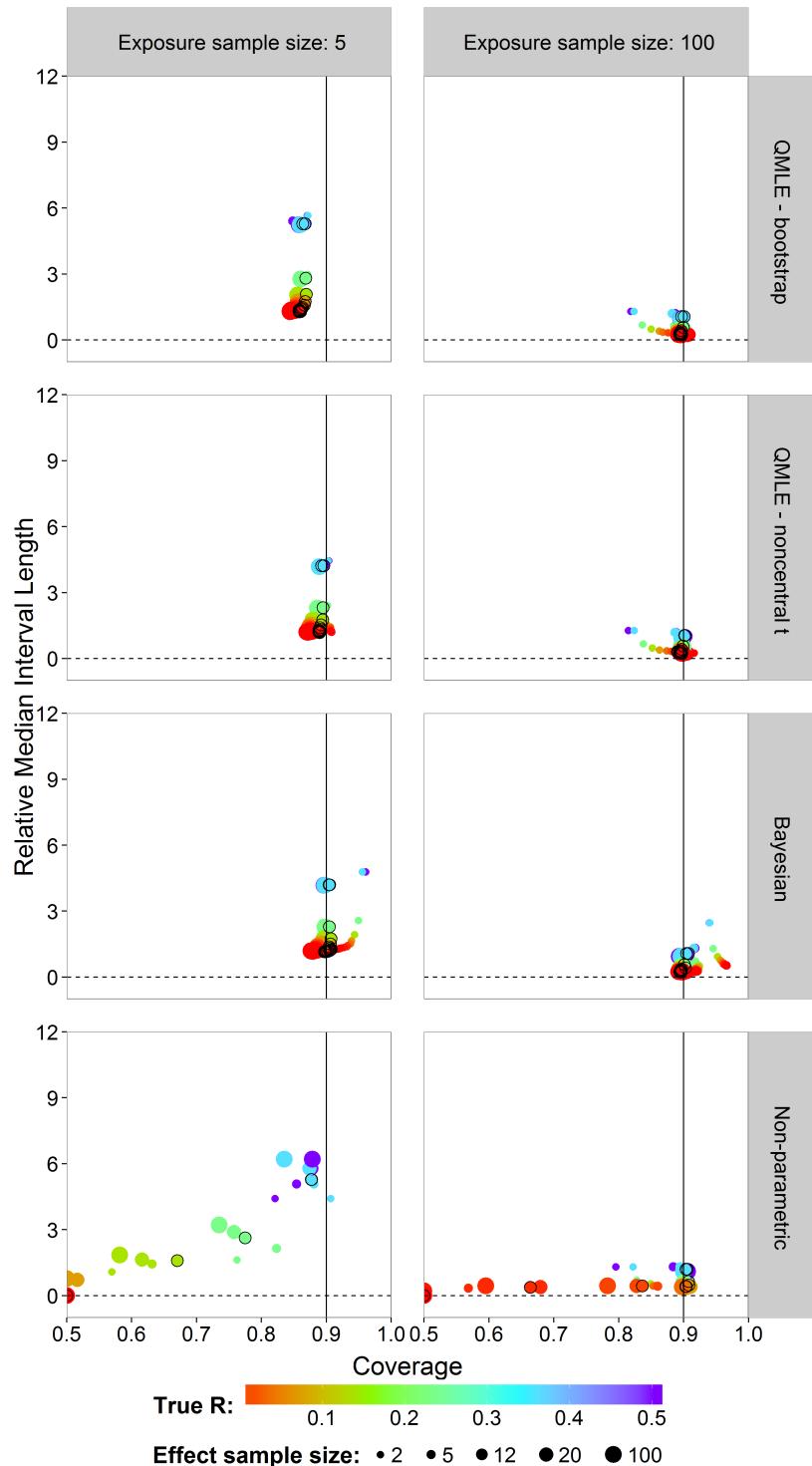
**Figure S16:** The median of the sampling distribution of the four point estimators,  $\hat{R}_{MLE}$ ,  $\hat{R}_{QMLE}$ ,  $\hat{R}_{Bayes}$  and  $\hat{R}_{NP}$  calculated on the probit scale. The diagonal dotted line represents the values where  $probit(\hat{R}) = probit(R)$ . The horizontal grey line gives the lower bound ( $probit\left(\frac{1}{n_x n_y + 2}\right)$ ) of  $probit(\hat{R}_{NP})$ .



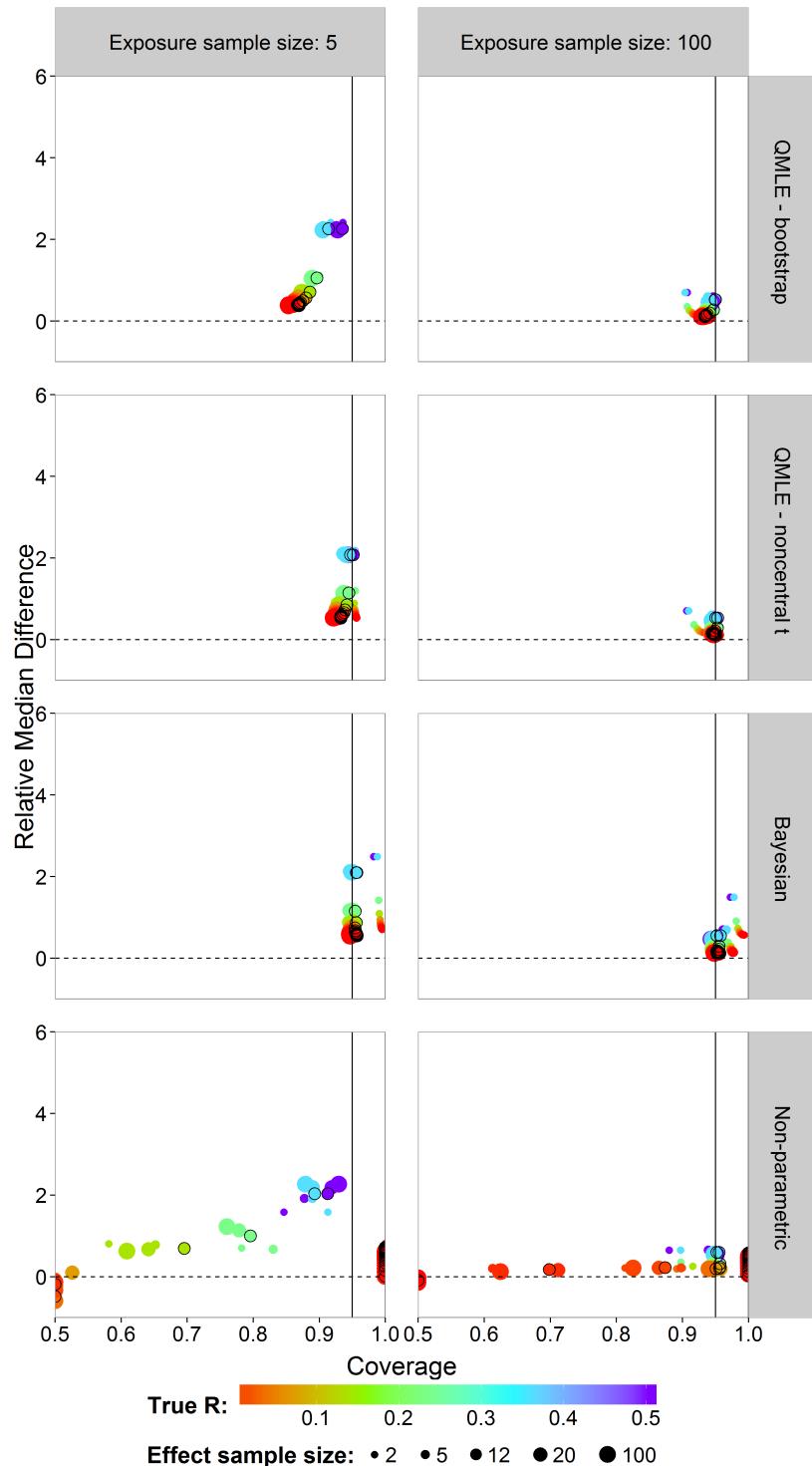
**Figure S17:** The 0.025 and 0.975 quantiles of the sampling distribution of the four point estimators,  $\hat{R}_{MLE}$ ,  $\hat{R}_{QMLE}$ ,  $\hat{R}_{Bayes}$  and  $\hat{R}_{NP}$  calculated on the probit scale. The diagonal dotted line represents the values where  $probit(\hat{R}) = probit(R)$ . The horizontal grey line gives the lower bound ( $probit\left(\frac{1}{nx ny + 2}\right)$ ) of  $probit(\hat{R}_{NP})$ .



**Figure S18:** Scatterplots of the 90% two-sided coverage probabilities against the relative median interval length calculated on the probit scale. The value of the true  $R$  value is illustrated by the color scale. The size of the dots corresponds to the size of the sample size of effect. A vertical reference line is drawn at a coverage probability of 90%. The points corresponding to  $n_y = 12$  are indicated by an open black circle.



**Figure S19:** Scatterplots of the 95% one-sided coverage probabilities against the relative median difference calculated on the probit scale. The value of the true  $R$  value is illustrated by the color scale. The size of the dots corresponds to the size of the sample size of effect. A vertical reference line is drawn at a coverage probability of 95%. The points corresponding to  $n_y = 12$  are indicated by a black circle.



**Table S6:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the MLE for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0026 (0.7379)	-0.0026	0.7379
	0.0744	-1.4438	-1.9999 (1.2222)	-0.5561	1.3428
	0.0006	-3.2486	-4.4966 (2.2665)	-1.2480	2.5874
	2.2e-07	-5.0534	-6.9932 (3.3927)	-1.9398	3.9081
	3.5e-12	-6.8582	-9.4898 (4.5403)	-2.6317	5.2478
(5,5)	0.5000	0.0000	-0.0141 (0.6726)	-0.0141	0.6728
	0.0744	-1.4438	-1.9827 (1.1013)	-0.5389	1.2261
	0.0006	-3.2486	-4.4435 (2.0338)	-1.1949	2.3589
	2.2e-07	-5.0534	-6.9043 (3.0417)	-1.8509	3.5606
	3.5e-12	-6.8582	-9.3651 (4.0694)	-2.5069	4.7796
(5,12)	0.5000	0.0000	0.0035 (0.6259)	0.0035	0.6259
	0.0744	-1.4438	-1.9206 (0.9807)	-0.4768	1.0904
	0.0006	-3.2486	-4.3258 (1.8348)	-1.0772	2.1276
	2.2e-07	-5.0534	-6.7310 (2.7661)	-1.6776	3.2350
	3.5e-12	-6.8582	-9.1361 (3.7171)	-2.2780	4.3595
(5,20)	0.5000	0.0000	0.0018 (0.6354)	0.0018	0.6354
	0.0744	-1.4438	-1.9312 (1.0372)	-0.4874	1.1460
	0.0006	-3.2486	-4.3474 (1.9211)	-1.0988	2.2131
	2.2e-07	-5.0534	-6.7636 (2.8770)	-1.7102	3.3469
	3.5e-12	-6.8582	-9.1798 (3.8517)	-2.3216	4.4973
(5,100)	0.5000	0.0000	0.0095 (0.6357)	0.0095	0.6357
	0.0744	-1.4438	-1.9328 (1.0125)	-0.4889	1.1244
	0.0006	-3.2486	-4.3606 (1.8890)	-1.1120	2.1920
	2.2e-07	-5.0534	-6.7885 (2.8416)	-1.7351	3.3294
	3.5e-12	-6.8582	-9.2163 (3.8137)	-2.3581	4.4839
(100,2)	0.5000	0.0000	-0.0008 (0.1720)	-0.0008	0.1720
	0.0744	-1.4438	-1.4773 (0.1997)	-0.0335	0.2024
	0.0006	-3.2486	-3.3230 (0.2906)	-0.0744	0.3000
	2.2e-07	-5.0534	-5.1687 (0.4053)	-0.1153	0.4214
	3.5e-12	-6.8582	-7.0144 (0.5286)	-0.1562	0.5512
(100,5)	0.5000	0.0000	0.0005 (0.1341)	0.0005	0.1341
	0.0744	-1.4438	-1.4654 (0.1690)	-0.0215	0.1704
	0.0006	-3.2486	-3.2977 (0.2675)	-0.0491	0.2719
	2.2e-07	-5.0534	-5.1301 (0.3841)	-0.0767	0.3917
	3.5e-12	-6.8582	-6.9625 (0.5065)	-0.1043	0.5172
(100,12)	0.5000	0.0000	0.0029 (0.1161)	0.0029	0.1161
	0.0744	-1.4438	-1.4601 (0.1559)	-0.0163	0.1568
	0.0006	-3.2486	-3.2889 (0.2583)	-0.0403	0.2615
	2.2e-07	-5.0534	-5.1177 (0.3758)	-0.0643	0.3813
	3.5e-12	-6.8582	-6.9465 (0.4978)	-0.0883	0.5056
(100,20)	0.5000	0.0000	-0.0015 (0.1078)	-0.0015	0.1079
	0.0744	-1.4438	-1.4653 (0.1500)	-0.0215	0.1516
	0.0006	-3.2486	-3.2952 (0.2543)	-0.0466	0.2585
	2.2e-07	-5.0534	-5.1250 (0.3722)	-0.0716	0.3790
	3.5e-12	-6.8582	-6.9549 (0.4941)	-0.0967	0.5035
(100,100)	0.5000	0.0000	-0.0031 (0.1016)	-0.0031	0.1016
	0.0744	-1.4438	-1.4642 (0.1440)	-0.0203	0.1454
	0.0006	-3.2486	-3.2905 (0.2489)	-0.0418	0.2524
	2.2e-07	-5.0534	-5.1167 (0.3668)	-0.0633	0.3722
	3.5e-12	-6.8582	-6.9430 (0.4883)	-0.0849	0.4956

**Table S7:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the QMLE for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0018 (0.6419)	-0.0018	0.6419
	0.0744	-1.4438	-1.7627 (1.0435)	-0.3189	1.0911
	0.0006	-3.2486	-3.9640 (1.9285)	-0.7154	2.0569
	2.2e-07	-5.0534	-6.1652 (2.8866)	-1.1118	3.0933
	3.5e-12	-6.8582	-8.3664 (3.8637)	-1.5082	4.1476
(5,5)	0.5000	0.0000	-0.0126 (0.6016)	-0.0126	0.6017
	0.0744	-1.4438	-1.7734 (0.9851)	-0.3296	1.0387
	0.0006	-3.2486	-3.9744 (1.8191)	-0.7258	1.9585
	2.2e-07	-5.0534	-6.1754 (2.7206)	-1.1220	2.9429
	3.5e-12	-6.8582	-8.3764 (3.6397)	-1.5182	3.9437
(5,12)	0.5000	0.0000	0.0033 (0.5645)	0.0033	0.5645
	0.0744	-1.4438	-1.7285 (0.8925)	-0.2847	0.9368
	0.0006	-3.2486	-3.8933 (1.6777)	-0.6447	1.7973
	2.2e-07	-5.0534	-6.0580 (2.5321)	-1.0046	2.7241
	3.5e-12	-6.8582	-8.2228 (3.4040)	-1.3646	3.6673
(5,20)	0.5000	0.0000	0.0014 (0.5748)	0.0014	0.5748
	0.0744	-1.4438	-1.7423 (0.9515)	-0.2985	0.9973
	0.0006	-3.2486	-3.9220 (1.7727)	-0.6734	1.8963
	2.2e-07	-5.0534	-6.1017 (2.6583)	-1.0483	2.8575
	3.5e-12	-6.8582	-8.2813 (3.5606)	-1.4232	3.8345
(5,100)	0.5000	0.0000	0.0086 (0.5777)	0.0086	0.5778
	0.0744	-1.4438	-1.7476 (0.9347)	-0.3038	0.9828
	0.0006	-3.2486	-3.9428 (1.7562)	-0.6942	1.8884
	2.2e-07	-5.0534	-6.1380 (2.6460)	-1.0846	2.8597
	3.5e-12	-6.8582	-8.3332 (3.5532)	-1.4751	3.8472
(100,2)	0.5000	0.0000	-0.0008 (0.1696)	-0.0008	0.1696
	0.0744	-1.4438	-1.4566 (0.1986)	-0.0128	0.1990
	0.0006	-3.2486	-3.2763 (0.2920)	-0.0277	0.2933
	2.2e-07	-5.0534	-5.0961 (0.4090)	-0.0427	0.4112
	3.5e-12	-6.8582	-6.9158 (0.5342)	-0.0576	0.5373
(100,5)	0.5000	0.0000	0.0005 (0.1329)	0.0005	0.1329
	0.0744	-1.4438	-1.4526 (0.1675)	-0.0087	0.1677
	0.0006	-3.2486	-3.2689 (0.2650)	-0.0203	0.2658
	2.2e-07	-5.0534	-5.0853 (0.3806)	-0.0319	0.3819
	3.5e-12	-6.8582	-6.9017 (0.5019)	-0.0435	0.5038
(100,12)	0.5000	0.0000	0.0029 (0.1153)	0.0029	0.1154
	0.0744	-1.4438	-1.4507 (0.1548)	-0.0069	0.1550
	0.0006	-3.2486	-3.2677 (0.2563)	-0.0191	0.2570
	2.2e-07	-5.0534	-5.0846 (0.3728)	-0.0312	0.3741
	3.5e-12	-6.8582	-6.9016 (0.4938)	-0.0434	0.4957
(100,20)	0.5000	0.0000	-0.0015 (0.1072)	-0.0015	0.1072
	0.0744	-1.4438	-1.4568 (0.1491)	-0.0130	0.1497
	0.0006	-3.2486	-3.2761 (0.2525)	-0.0275	0.2540
	2.2e-07	-5.0534	-5.0953 (0.3696)	-0.0419	0.3720
	3.5e-12	-6.8582	-6.9145 (0.4906)	-0.0563	0.4938
(100,100)	0.5000	0.0000	-0.0031 (0.1011)	-0.0031	0.1011
	0.0744	-1.4438	-1.4568 (0.1432)	-0.0130	0.1438
	0.0006	-3.2486	-3.2740 (0.2476)	-0.0253	0.2489
	2.2e-07	-5.0534	-5.0911 (0.3649)	-0.0377	0.3669
	3.5e-12	-6.8582	-6.9082 (0.4859)	-0.0501	0.4884

**Table S8:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the Bayes estimator for all sample size scenarios and selected  $R$  values.

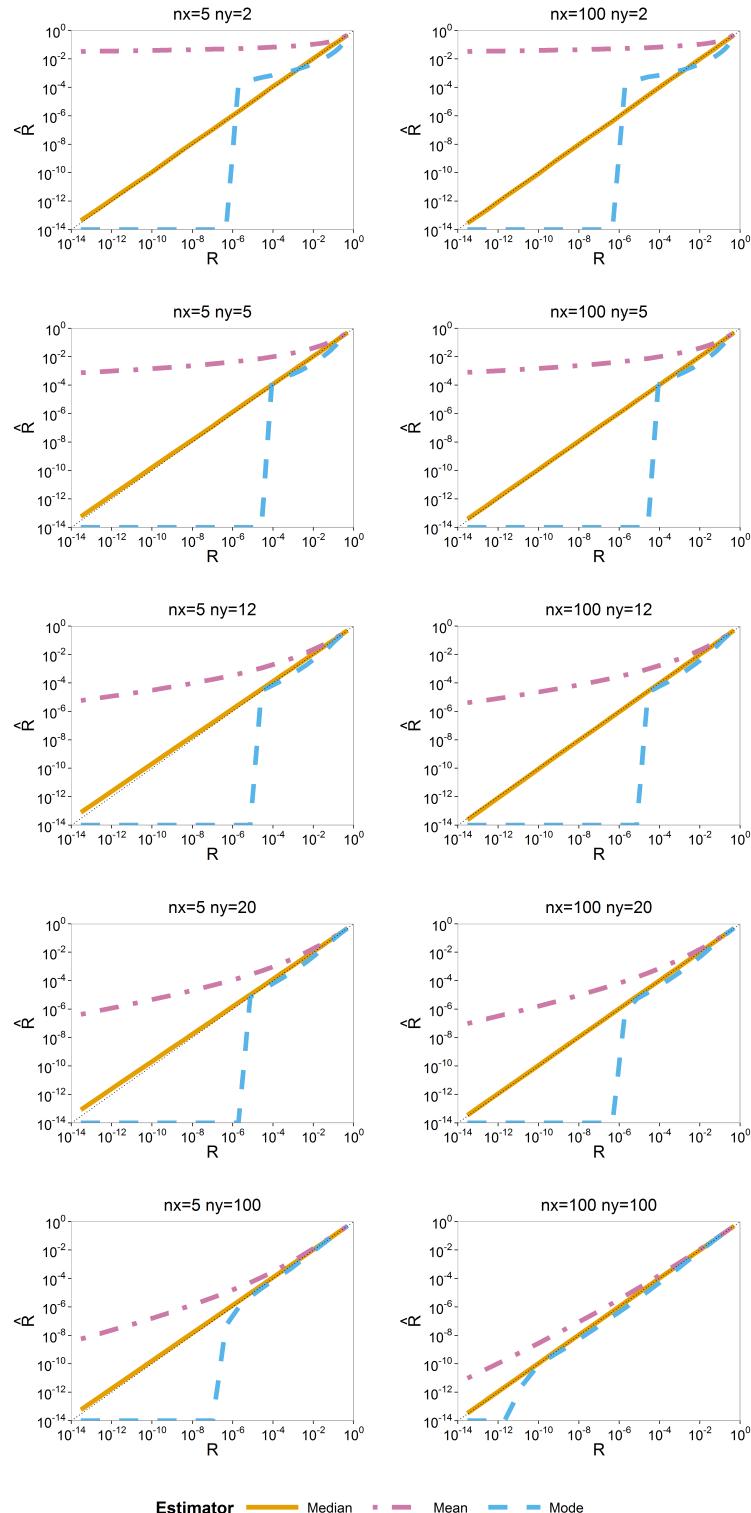
$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0010 (0.5235)	-0.0010	0.5235
	0.0744	-1.4438	-1.4531 (0.8106)	-0.0092	0.8106
	0.0006	-3.2486	-3.2427 (1.4823)	0.0059	1.4823
	2.2e-07	-5.0534	-5.0336 (2.2163)	0.0198	2.2164
	3.5e-12	-6.8582	-6.8258 (2.9658)	0.0323	2.9660
(5,5)	0.5000	0.0000	-0.0111 (0.5406)	-0.0111	0.5407
	0.0744	-1.4438	-1.5968 (0.8564)	-0.1529	0.8699
	0.0006	-3.2486	-3.5638 (1.5670)	-0.3151	1.5984
	2.2e-07	-5.0534	-5.5314 (2.3400)	-0.4780	2.3883
	3.5e-12	-6.8582	-7.4998 (3.1289)	-0.6416	3.1941
(5,12)	0.5000	0.0000	0.0030 (0.5208)	0.0030	0.5208
	0.0744	-1.4438	-1.5901 (0.8090)	-0.1463	0.8221
	0.0006	-3.2486	-3.5658 (1.5191)	-0.3172	1.5518
	2.2e-07	-5.0534	-5.5420 (2.2940)	-0.4886	2.3455
	3.5e-12	-6.8582	-7.5193 (3.0847)	-0.6611	3.1548
(5,20)	0.5000	0.0000	0.0014 (0.5337)	0.0014	0.5337
	0.0744	-1.4438	-1.6105 (0.8737)	-0.1666	0.8894
	0.0006	-3.2486	-3.6087 (1.6309)	-0.3601	1.6702
	2.2e-07	-5.0534	-5.6073 (2.4488)	-0.5539	2.5106
	3.5e-12	-6.8582	-7.6071 (3.2816)	-0.7489	3.3660
(5,100)	0.5000	0.0000	0.0083 (0.5405)	0.0083	0.5405
	0.0744	-1.4438	-1.6231 (0.8713)	-0.1792	0.8896
	0.0006	-3.2486	-3.6445 (1.6452)	-0.3959	1.6922
	2.2e-07	-5.0534	-5.6665 (2.4835)	-0.6131	2.5581
	3.5e-12	-6.8582	-7.6897 (3.3373)	-0.8315	3.4393
(100,2)	0.5000	0.0000	-0.0007 (0.1626)	-0.0007	0.1626
	0.0744	-1.4438	-1.3939 (0.1998)	0.0499	0.2060
	0.0006	-3.2486	-3.1335 (0.3099)	0.1151	0.3305
	2.2e-07	-5.0534	-4.8727 (0.4417)	0.1807	0.4772
	3.5e-12	-6.8582	-6.6116 (0.5810)	0.2465	0.6311
(100,5)	0.5000	0.0000	0.0005 (0.1311)	0.0005	0.1311
	0.0744	-1.4438	-1.4320 (0.1652)	0.0118	0.1656
	0.0006	-3.2486	-3.2201 (0.2616)	0.0285	0.2632
	2.2e-07	-5.0534	-5.0078 (0.3760)	0.0456	0.3788
	3.5e-12	-6.8582	-6.7955 (0.4961)	0.0627	0.5001
(100,12)	0.5000	0.0000	0.0028 (0.1146)	0.0028	0.1146
	0.0744	-1.4438	-1.4401 (0.1535)	0.0037	0.1535
	0.0006	-3.2486	-3.2434 (0.2537)	0.0052	0.2537
	2.2e-07	-5.0534	-5.0467 (0.3688)	0.0067	0.3689
	3.5e-12	-6.8582	-6.8499 (0.4884)	0.0083	0.4885
(100,20)	0.5000	0.0000	-0.0015 (0.1066)	-0.0015	0.1067
	0.0744	-1.4438	-1.4489 (0.1480)	-0.0051	0.1481
	0.0006	-3.2486	-3.2580 (0.2504)	-0.0094	0.2506
	2.2e-07	-5.0534	-5.0671 (0.3664)	-0.0137	0.3666
	3.5e-12	-6.8582	-6.8762 (0.4862)	-0.0180	0.4866
(100,100)	0.5000	0.0000	-0.0031 (0.1008)	-0.0031	0.1008
	0.0744	-1.4438	-1.4519 (0.1427)	-0.0080	0.1429
	0.0006	-3.2486	-3.2623 (0.2467)	-0.0136	0.2471
	2.2e-07	-5.0534	-5.0726 (0.3635)	-0.0193	0.3641
	3.5e-12	-6.8582	-6.8831 (0.4840)	-0.0249	0.4847

**Table S9:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the non-parametric estimator for all sample size scenarios and selected  $R$  values.

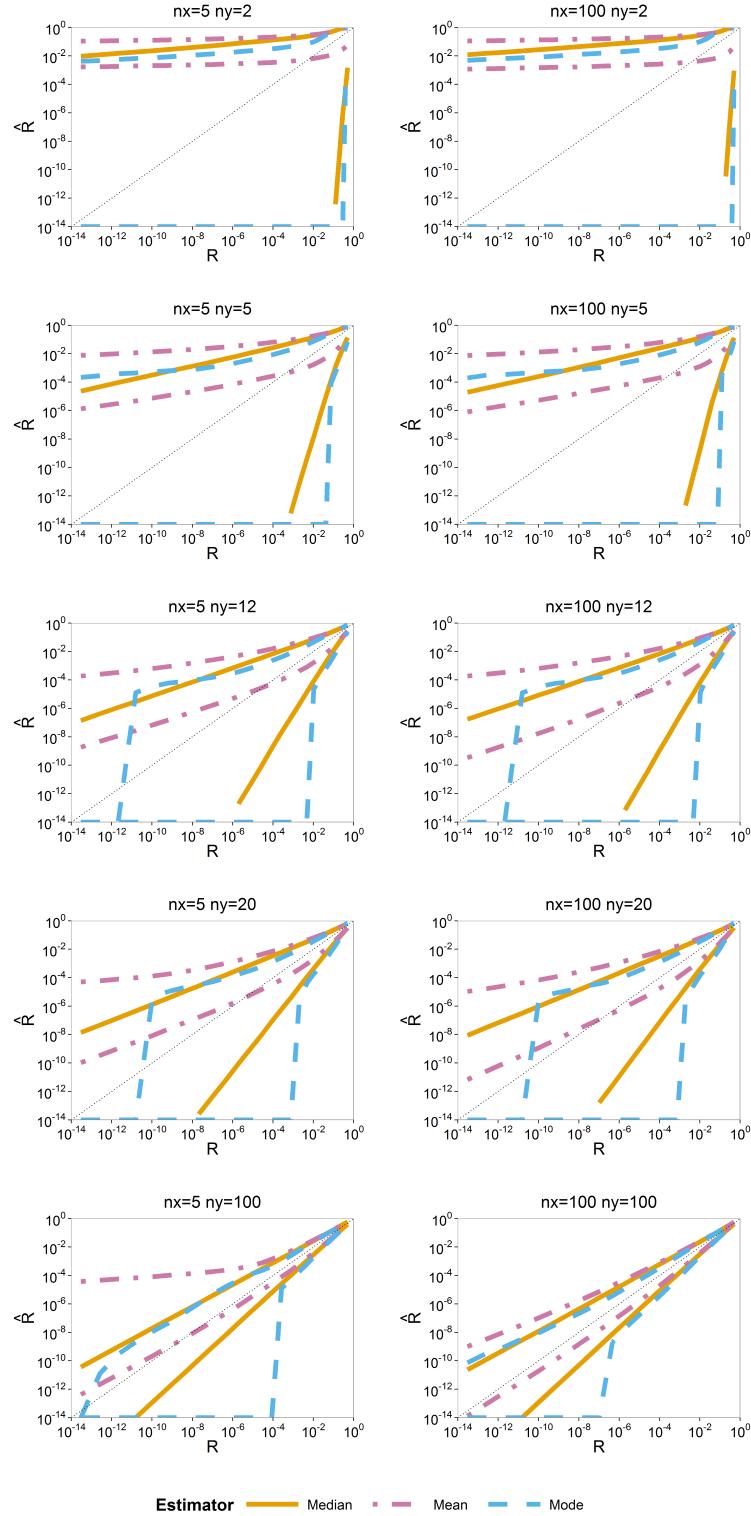
$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	0.0024 (0.5273)	0.0024	0.5273
	0.0744	-1.4438	-1.1263 (0.3676)	0.3176	0.4858
	0.0006	-3.2486	-1.3808 (0.0371)	1.8678	1.8682
	2.2e-07	-5.0534	-1.3830 (0.0000)	3.6704	3.6704
	3.5e-12	-6.8582	-1.3830 (0.0000)	5.4752	5.4752
(5,5)	0.5000	0.0000	-0.0109 (0.5885)	-0.0109	0.5886
	0.0744	-1.4438	-1.4070 (0.4858)	0.0368	0.4872
	0.0006	-3.2486	-1.7833 (0.0416)	1.4653	1.4659
	2.2e-07	-5.0534	-1.7862 (0.0000)	3.2672	3.2672
	3.5e-12	-6.8582	-1.7862 (0.0000)	5.0720	5.0720
(5,12)	0.5000	0.0000	0.0023 (0.6123)	0.0023	0.6123
	0.0744	-1.4438	-1.6243 (0.6131)	-0.1805	0.6391
	0.0006	-3.2486	-2.1354 (0.0716)	1.1132	1.1155
	2.2e-07	-5.0534	-2.1412 (0.0000)	2.9122	2.9122
	3.5e-12	-6.8582	-2.1412 (0.0000)	4.7170	4.7170
(5,20)	0.5000	0.0000	0.0047 (0.6235)	0.0047	0.6235
	0.0744	-1.4438	-1.7203 (0.6916)	-0.2765	0.7448
	0.0006	-3.2486	-2.3272 (0.0819)	0.9214	0.9251
	2.2e-07	-5.0534	-2.3338 (0.0000)	2.7196	2.7196
	3.5e-12	-6.8582	-2.3338 (0.0000)	4.5244	4.5244
(5,100)	0.5000	0.0000	0.0091 (0.6515)	0.0091	0.6516
	0.0744	-1.4438	-1.9859 (0.9037)	-0.5421	1.0538
	0.0006	-3.2486	-2.8670 (0.1244)	0.3816	0.4014
	2.2e-07	-5.0534	-2.8790 (0.0277)	2.1744	2.1746
	3.5e-12	-6.8582	-2.8794 (0.0000)	3.9788	3.9788
(100,2)	0.5000	0.0000	-0.0032 (0.1811)	-0.0032	0.1811
	0.0744	-1.4438	-1.4490 (0.2257)	-0.0052	0.2257
	0.0006	-3.2486	-2.5535 (0.0914)	0.6951	0.7011
	2.2e-07	-5.0534	-2.5793 (0.0000)	2.4741	2.4741
	3.5e-12	-6.8582	-2.5793 (0.0000)	4.2789	4.2789
(100,5)	0.5000	0.0000	0.0006 (0.1465)	0.0006	0.1465
	0.0744	-1.4438	-1.4577 (0.1993)	-0.0138	0.1998
	0.0006	-3.2486	-2.8367 (0.1363)	0.4119	0.4338
	2.2e-07	-5.0534	-2.8794 (0.0000)	2.1740	2.1740
	3.5e-12	-6.8582	-2.8794 (0.0000)	3.9788	3.9788
(100,12)	0.5000	0.0000	0.0033 (0.1291)	0.0033	0.1291
	0.0744	-1.4438	-1.4581 (0.1865)	-0.0143	0.1871
	0.0006	-3.2486	-3.0798 (0.1867)	0.1688	0.2517
	2.2e-07	-5.0534	-3.1445 (0.0000)	1.9089	1.9089
	3.5e-12	-6.8582	-3.1445 (0.0000)	3.7137	3.7137
(100,20)	0.5000	0.0000	-0.0004 (0.1227)	-0.0004	0.1227
	0.0744	-1.4438	-1.4686 (0.1804)	-0.0248	0.1821
	0.0006	-3.2486	-3.2080 (0.2246)	0.0406	0.2283
	2.2e-07	-5.0534	-3.2908 (0.0000)	1.7626	1.7626
	3.5e-12	-6.8582	-3.2908 (0.0000)	3.5674	3.5674
(100,100)	0.5000	0.0000	-0.0025 (0.1153)	-0.0025	0.1154
	0.0744	-1.4438	-1.4701 (0.1802)	-0.0262	0.1821
	0.0006	-3.2486	-3.5579 (0.3598)	-0.3093	0.4744
	2.2e-07	-5.0534	-3.7190 (0.0027)	1.3344	1.3344
	3.5e-12	-6.8582	-3.7191 (0.0000)	3.1391	3.1391

## 2.4 Case: $\sigma_y = 5\sigma_x$

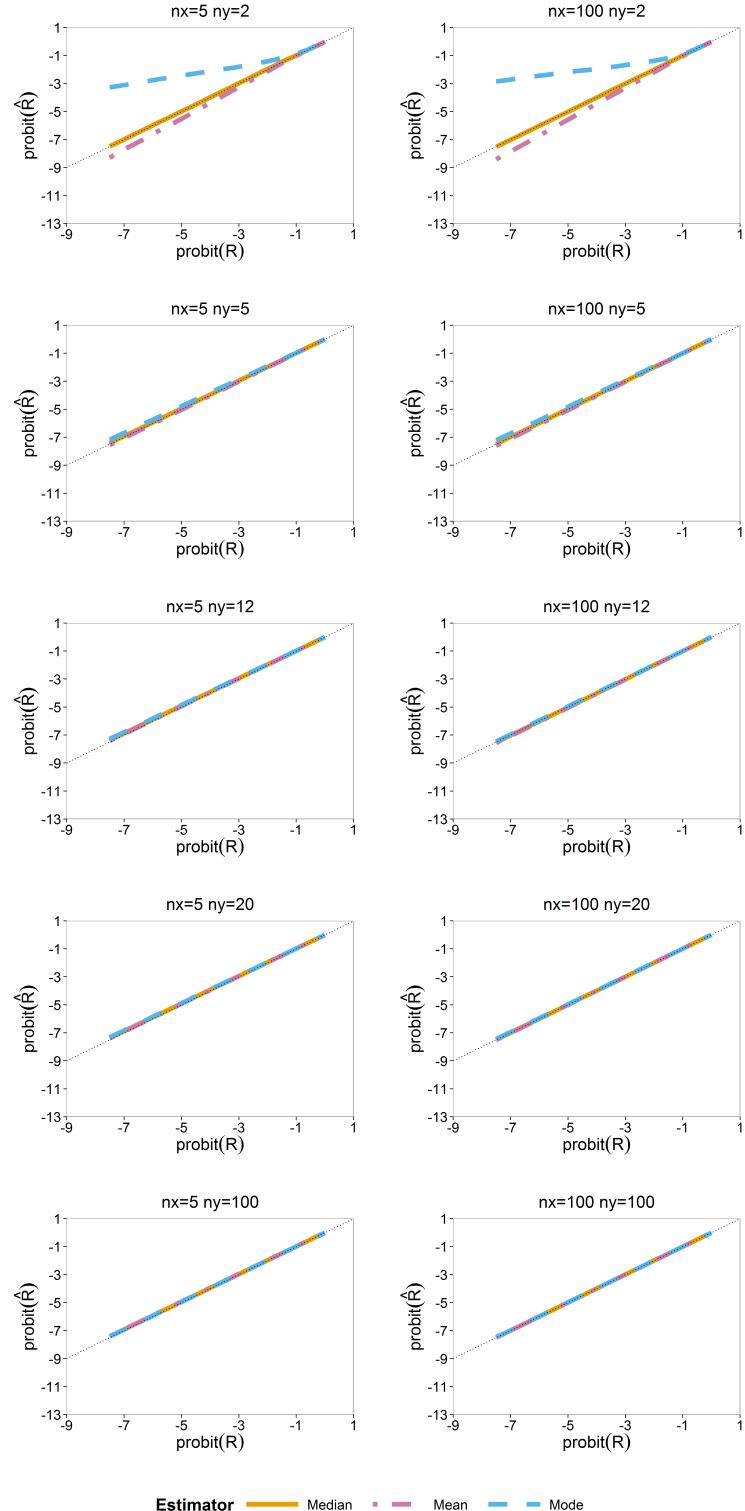
**Figure S20:** The median of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the original scale and plotted on  $\log_{10}$ -scale. When plotted on  $\log_{10}$ -scale, a zero mode becomes  $-\infty$ . The diagonal dotted line represents the values where  $\hat{R} = R$ .



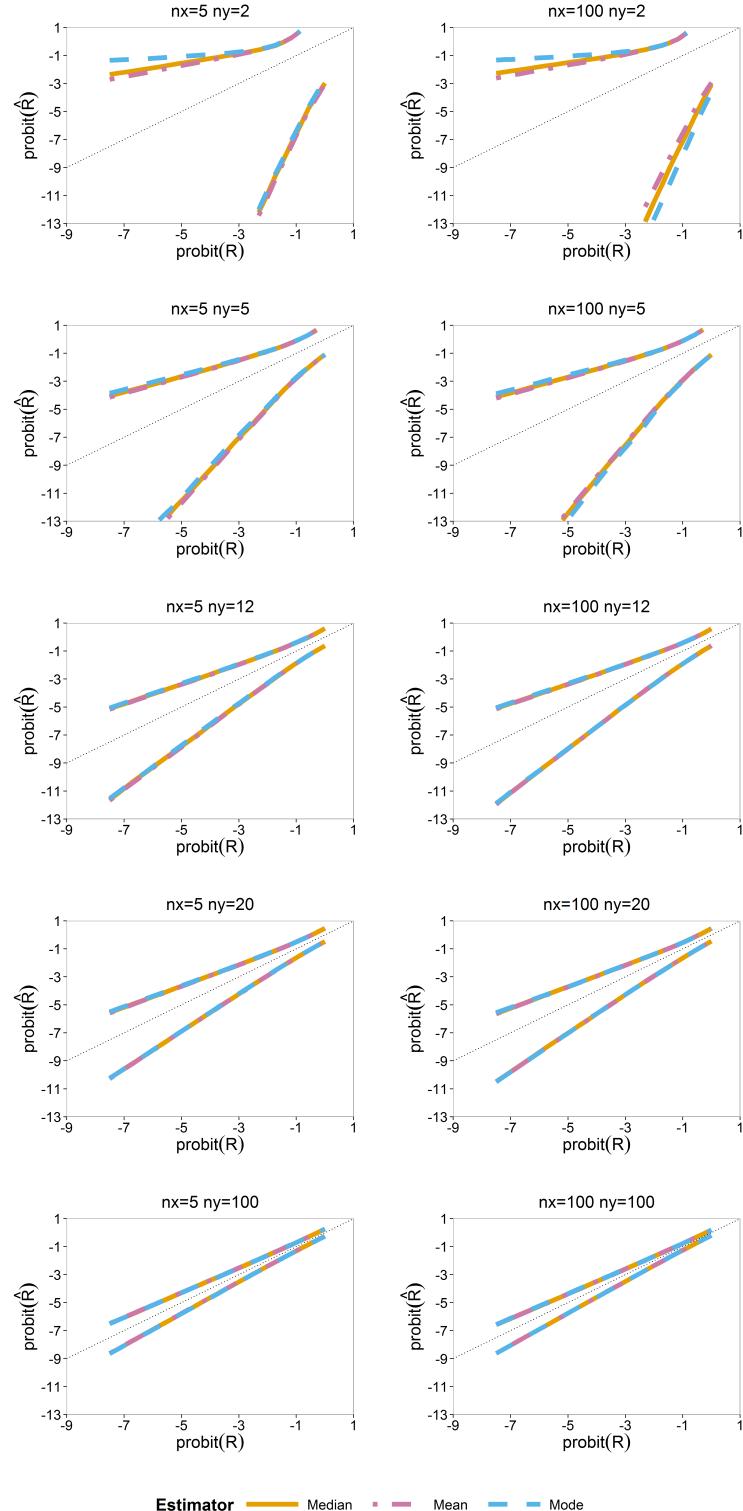
**Figure S21:** The 0.025 and 0.975 quantiles of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the original scale and plotted on  $\log_{10}$ -scale. When plotted on  $\log_{10}$ -scale, a zero mode becomes  $-\infty$ . The diagonal dotted line represents the values where  $\hat{R} = R$ .



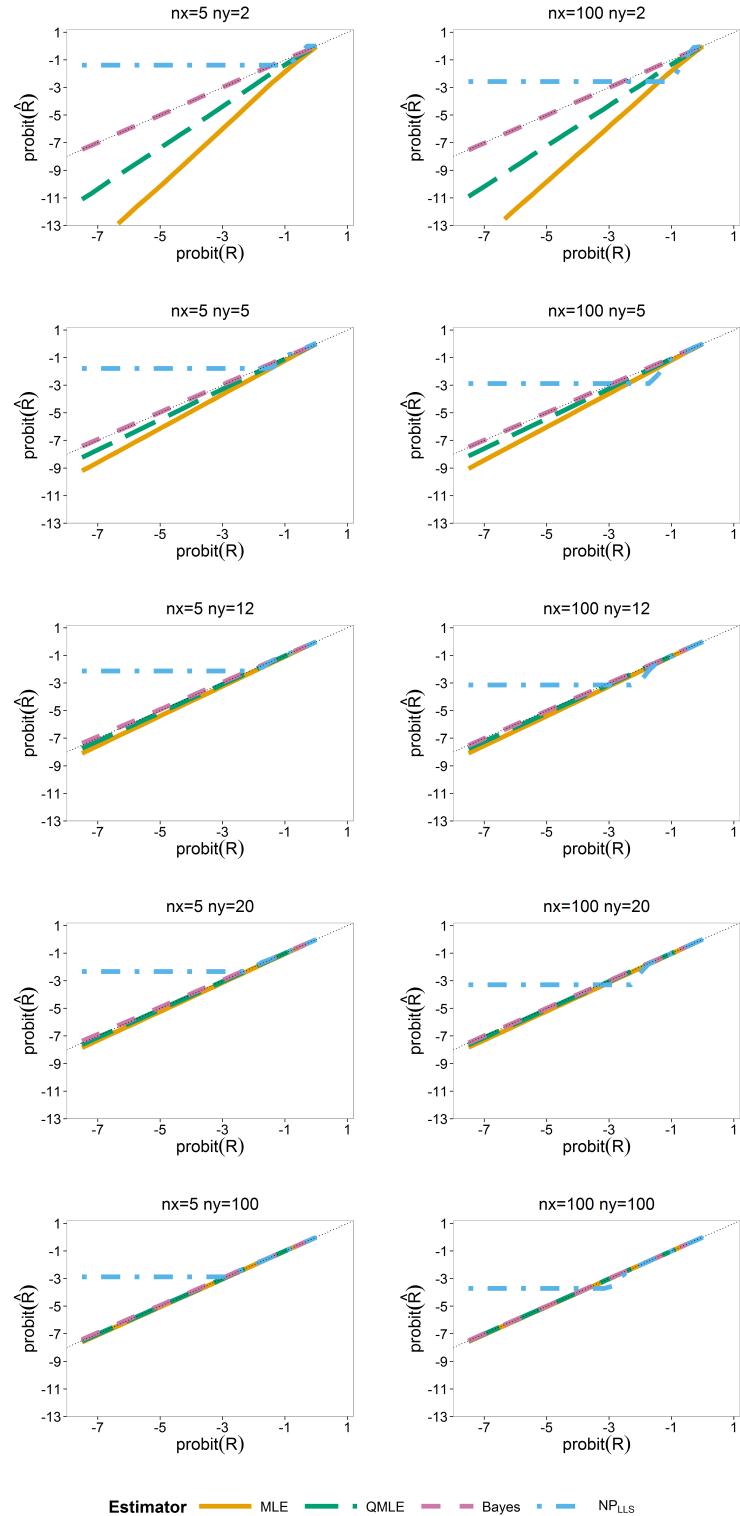
**Figure S22:** The median of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the probit scale. The diagonal dotted line represents the values where  $\text{probit}(\hat{R}) = \text{probit}(R)$ .



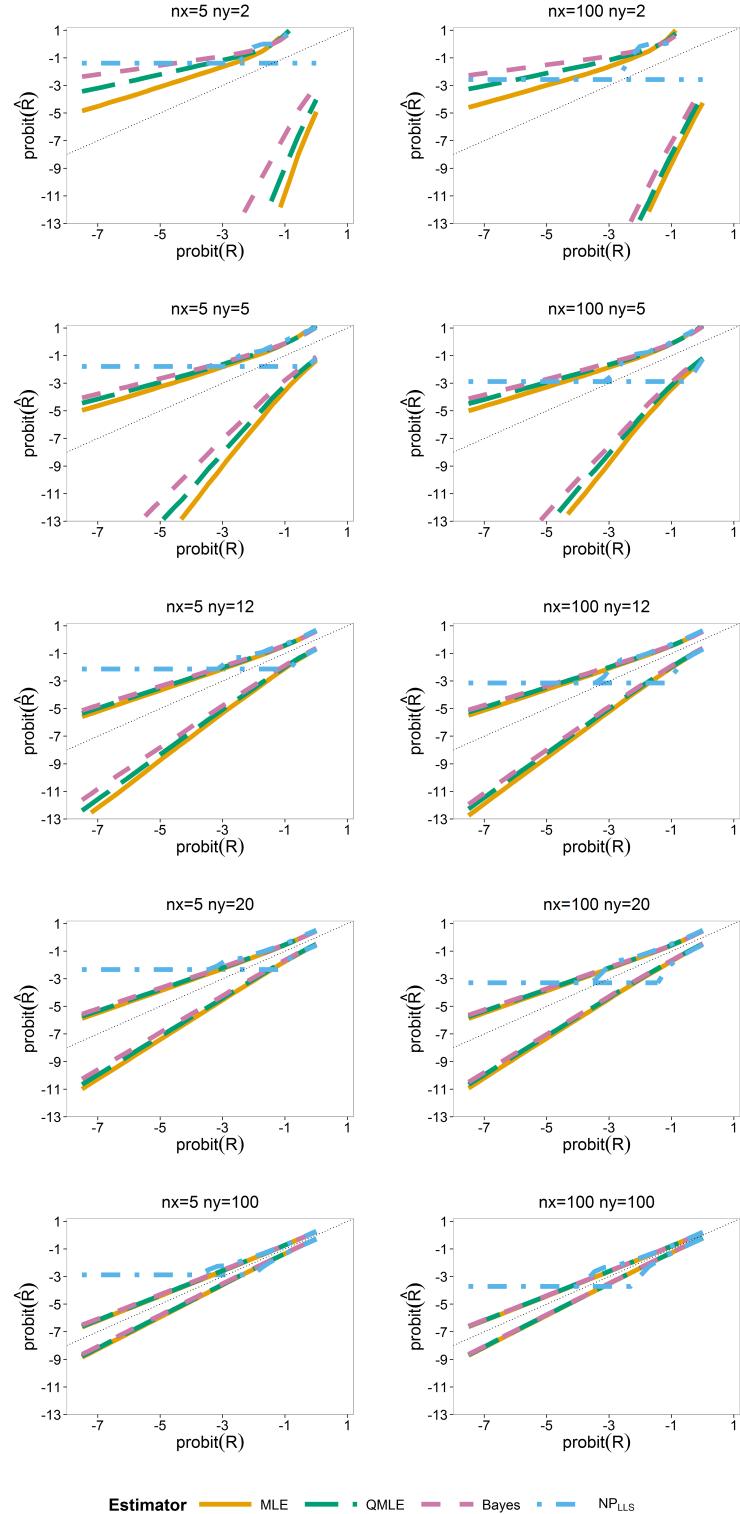
**Figure S23:** The 0.025 and 0.975 quantiles of the sampling distribution of the three Bayesian point estimators (mean, median and mode) calculated on the probit scale. The diagonal dotted line represents the values where  $\text{probit}(\hat{R}) = \text{probit}(R)$ .



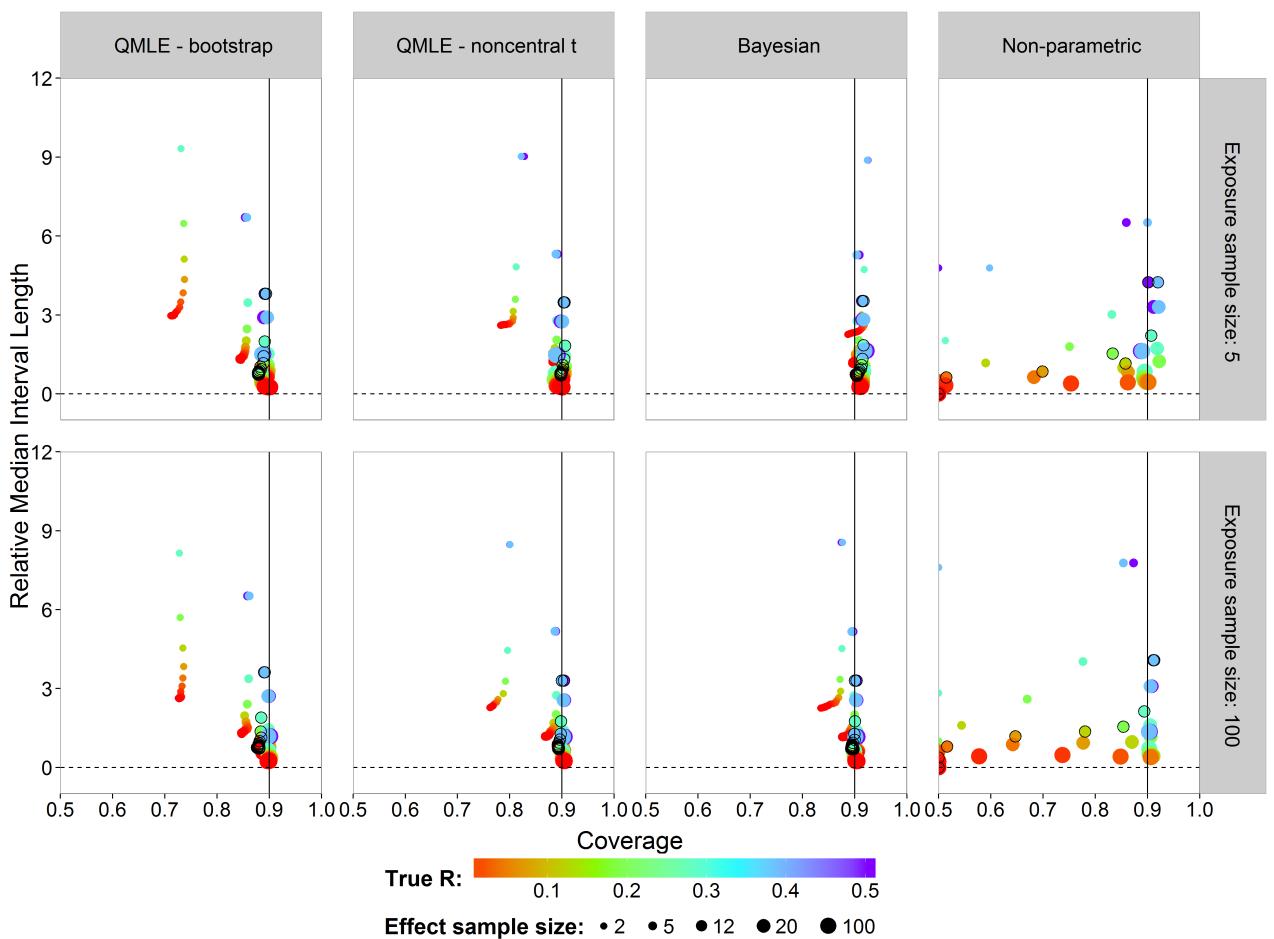
**Figure S24:** The median of the sampling distribution of the four point estimators,  $\hat{R}_{MLE}$ ,  $\hat{R}_{QMLE}$ ,  $\hat{R}_{Bayes}$  and  $\hat{R}_{NP}$  calculated on the probit scale. The diagonal dotted line represents the values where  $probit(\hat{R}) = probit(R)$ . The horizontal grey line gives the lower bound ( $probit\left(\frac{1}{n_x n_y + 2}\right)$ ) of  $probit(\hat{R}_{NP})$ .



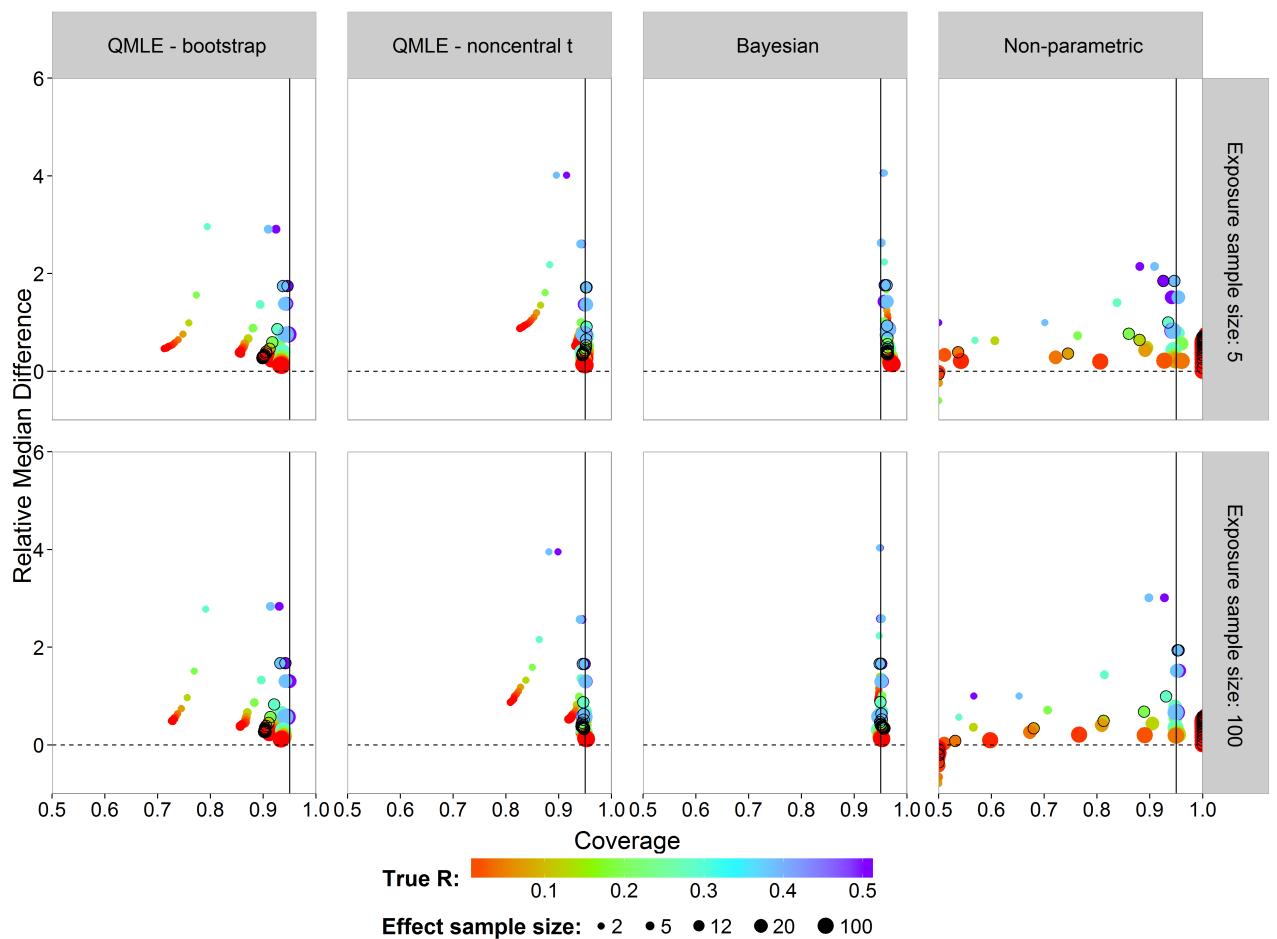
**Figure S25:** The 0.025 and 0.975 quantiles of the sampling distribution of the four point estimators,  $\hat{R}_{MLE}$ ,  $\hat{R}_{QMLE}$ ,  $\hat{R}_{Bayes}$  and  $\hat{R}_{NP}$  calculated on the probit scale. The diagonal dotted line represents the values where  $probit(\hat{R}) = probit(R)$ . The horizontal grey line gives the lower bound ( $probit\left(\frac{1}{nx ny + 2}\right)$ ) of  $probit(\hat{R}_{NP})$ .



**Figure S26:** Scatterplots of the 90% two-sided coverage probabilities against the relative median interval length calculated on the probit scale. The value of the true  $R$  value is illustrated by the color scale. The size of the dots corresponds to the size of the sample size of effect. A vertical reference line is drawn at a coverage probability of 90%. The points corresponding to  $n_y = 12$  are indicated by an open black circle.



**Figure S27:** Scatterplots of the 95% one-sided coverage probabilities against the relative median difference calculated on the probit scale. The value of the true  $R$  value is illustrated by the color scale. The size of the dots corresponds to the size of the sample size of effect. A vertical reference line is drawn at a coverage probability of 95%. The points corresponding to  $n_y = 12$  are indicated by a black circle.



**Table S10:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the MLE for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0307 (2.4070)	-0.0307	2.4072
	0.1240	-1.1551	-3.0883 (3.2728)	-1.9333	3.8012
	0.0047	-2.5989	-6.9104 (5.5870)	-4.3115	7.0571
	2.6e-05	-4.0427	-10.7324 (8.2261)	-6.6897	10.6028
	2.0e-08	-5.4865	-14.5545 (10.9578)	-9.0679	14.2232
	2.1e-12	-6.9304	-18.3765 (13.7270)	-11.4461	17.8730
(5,5)	0.5000	0.0000	-0.0019 (0.6570)	-0.0019	0.6570
	0.1240	-1.1551	-1.5759 (0.9441)	-0.4208	1.0337
	0.0047	-2.5989	-3.5433 (1.6449)	-0.9444	1.8968
	2.6e-05	-4.0427	-5.5108 (2.4286)	-1.4681	2.8378
	2.0e-08	-5.4865	-7.4783 (3.2353)	-1.9917	3.7993
	2.1e-12	-6.9304	-9.4457 (4.0515)	-2.5153	4.7688
(5,12)	0.5000	0.0000	-0.0011 (0.3369)	-0.0011	0.3369
	0.1240	-1.1551	-1.2954 (0.4476)	-0.1404	0.4691
	0.0047	-2.5989	-2.9134 (0.7443)	-0.3145	0.8080
	2.6e-05	-4.0427	-4.5314 (1.0862)	-0.4887	1.1911
	2.0e-08	-5.4865	-6.1493 (1.4415)	-0.6628	1.5866
	2.1e-12	-6.9304	-7.7673 (1.8024)	-0.8369	1.9872
(5,20)	0.5000	0.0000	-0.0012 (0.2560)	-0.0012	0.2560
	0.1240	-1.1551	-1.2308 (0.3235)	-0.0757	0.3323
	0.0047	-2.5989	-2.7678 (0.5181)	-0.1689	0.5449
	2.6e-05	-4.0427	-4.3048 (0.7482)	-0.2620	0.7928
	2.0e-08	-5.4865	-5.8417 (0.9893)	-0.3552	1.0512
	2.1e-12	-6.9304	-7.3787 (1.2350)	-0.4484	1.3139
(5,100)	0.5000	0.0000	0.0021 (0.1341)	0.0021	0.1341
	0.1240	-1.1551	-1.1728 (0.1603)	-0.0177	0.1613
	0.0047	-2.5989	-2.6414 (0.2338)	-0.0425	0.2376
	2.6e-05	-4.0427	-4.1100 (0.3241)	-0.0673	0.3310
	2.0e-08	-5.4865	-5.5786 (0.4206)	-0.0921	0.4306
	2.1e-12	-6.9304	-7.0472 (0.5198)	-0.1168	0.5328
(100,2)	0.5000	0.0000	-0.0377 (1.8901)	-0.0377	1.8905
	0.1240	-1.1551	-2.7841 (2.4844)	-1.6290	2.9708
	0.0047	-2.5989	-6.2170 (4.0743)	-3.6181	5.4489
	2.6e-05	-4.0427	-9.6499 (5.9165)	-5.6072	8.1514
	2.0e-08	-5.4865	-13.0828 (7.8348)	-7.5963	10.9128
	2.1e-12	-6.9304	-16.5157 (9.7847)	-9.5854	13.6975
(100,5)	0.5000	0.0000	0.0036 (0.6321)	0.0036	0.6321
	0.1240	-1.1551	-1.5439 (0.8846)	-0.3889	0.9663
	0.0047	-2.5989	-3.4783 (1.5562)	-0.8794	1.7875
	2.6e-05	-4.0427	-5.4127 (2.3131)	-1.3700	2.6884
	2.0e-08	-5.4865	-7.3470 (3.0933)	-1.8605	3.6097
	2.1e-12	-6.9304	-9.2814 (3.8828)	-2.3510	4.5391
(100,12)	0.5000	0.0000	-0.0015 (0.3186)	-0.0015	0.3186
	0.1240	-1.1551	-1.2845 (0.4293)	-0.1294	0.4484
	0.0047	-2.5989	-2.8883 (0.7150)	-0.2894	0.7713
	2.6e-05	-4.0427	-4.4920 (1.0422)	-0.4493	1.1350
	2.0e-08	-5.4865	-6.0958 (1.3819)	-0.6092	1.5102
	2.1e-12	-6.9304	-7.6995 (1.7266)	-0.7691	1.8901
(100,20)	0.5000	0.0000	0.0037 (0.2356)	0.0037	0.2356
	0.1240	-1.1551	-1.2256 (0.3101)	-0.0706	0.3180
	0.0047	-2.5989	-2.7623 (0.5067)	-0.1634	0.5324
	2.6e-05	-4.0427	-4.2990 (0.7344)	-0.2563	0.7778
	2.0e-08	-5.4865	-5.8357 (0.9716)	-0.3491	1.0324
	2.1e-12	-6.9304	-7.3724 (1.2126)	-0.4420	1.2907
(100,100)	0.5000	0.0000	-0.0024 (0.1003)	-0.0024	0.1004
	0.1240	-1.1551	-1.1699 (0.1286)	-0.0148	0.1294
	0.0047	-2.5989	-2.6292 (0.2049)	-0.0303	0.2071
	2.6e-05	-4.0427	-4.0886 (0.2945)	-0.0459	0.2981
	2.0e-08	-5.4865	-5.5479 (0.3883)	-0.0614	0.3932
	2.1e-12	-6.9304	-7.0073 (0.4839)	-0.0769	0.4900

**Table S11:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the QMLE for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0215 (1.9343)	-0.0215	1.9344
	0.1240	-1.1551	-2.3912 (2.7028)	-1.2362	2.9721
	0.0047	-2.5989	-5.3534 (4.7159)	-2.7545	5.4615
	2.6e-05	-4.0427	-8.3156 (6.9871)	-4.2729	8.1901
	2.0e-08	-5.4865	-11.2779 (9.3298)	-5.7913	10.9811
	2.1e-12	-6.9304	-14.2401 (11.7010)	-7.3097	13.7966
(5,5)	0.5000	0.0000	-0.0017 (0.5877)	-0.0017	0.5877
	0.1240	-1.1551	-1.4095 (0.8445)	-0.2544	0.8820
	0.0047	-2.5989	-3.1693 (1.4713)	-0.5704	1.5780
	2.6e-05	-4.0427	-4.9290 (2.1722)	-0.8863	2.3460
	2.0e-08	-5.4865	-6.6888 (2.8938)	-1.2022	3.1336
	2.1e-12	-6.9304	-8.4485 (3.6238)	-1.5181	3.9289
(5,12)	0.5000	0.0000	-0.0010 (0.3215)	-0.0010	0.3215
	0.1240	-1.1551	-1.2364 (0.4259)	-0.0813	0.4335
	0.0047	-2.5989	-2.7805 (0.7064)	-0.1817	0.7294
	2.6e-05	-4.0427	-4.3247 (1.0303)	-0.2820	1.0682
	2.0e-08	-5.4865	-5.8689 (1.3669)	-0.3824	1.4194
	2.1e-12	-6.9304	-7.4131 (1.7088)	-0.4827	1.7757
(5,20)	0.5000	0.0000	-0.0012 (0.2486)	-0.0012	0.2487
	0.1240	-1.1551	-1.1955 (0.3135)	-0.0404	0.3161
	0.0047	-2.5989	-2.6884 (0.5010)	-0.0895	0.5090
	2.6e-05	-4.0427	-4.1814 (0.7230)	-0.1386	0.7362
	2.0e-08	-5.4865	-5.6743 (0.9557)	-0.1877	0.9740
	2.1e-12	-6.9304	-7.1672 (1.1929)	-0.2368	1.2162
(5,100)	0.5000	0.0000	0.0021 (0.1329)	0.0021	0.1329
	0.1240	-1.1551	-1.1626 (0.1588)	-0.0075	0.1590
	0.0047	-2.5989	-2.6184 (0.2316)	-0.0195	0.2324
	2.6e-05	-4.0427	-4.0742 (0.3211)	-0.0315	0.3227
	2.0e-08	-5.4865	-5.5300 (0.4167)	-0.0435	0.4190
	2.1e-12	-6.9304	-6.9858 (0.5150)	-0.0555	0.5180
(100,2)	0.5000	0.0000	-0.0292 (1.6350)	-0.0292	1.6352
	0.1240	-1.1551	-2.2766 (2.2785)	-1.1216	2.5396
	0.0047	-2.5989	-5.0859 (3.9147)	-2.4870	4.6379
	2.6e-05	-4.0427	-7.8952 (5.7636)	-3.8524	6.9325
	2.0e-08	-5.4865	-10.7044 (7.6729)	-5.2179	9.2790
	2.1e-12	-6.9304	-13.5137 (9.6068)	-6.5833	11.6461
(100,5)	0.5000	0.0000	0.0034 (0.5742)	0.0034	0.5742
	0.1240	-1.1551	-1.3960 (0.8145)	-0.2410	0.8494
	0.0047	-2.5989	-3.1453 (1.4467)	-0.5465	1.5465
	2.6e-05	-4.0427	-4.8946 (2.1558)	-0.8519	2.3180
	2.0e-08	-5.4865	-6.6439 (2.8856)	-1.1574	3.1091
	2.1e-12	-6.9304	-8.3932 (3.6237)	-1.4628	3.9078
(100,12)	0.5000	0.0000	-0.0015 (0.3057)	-0.0015	0.3057
	0.1240	-1.1551	-1.2323 (0.4127)	-0.0772	0.4199
	0.0047	-2.5989	-2.7709 (0.6885)	-0.1720	0.7097
	2.6e-05	-4.0427	-4.3094 (1.0042)	-0.2667	1.0390
	2.0e-08	-5.4865	-5.8480 (1.3317)	-0.3615	1.3799
	2.1e-12	-6.9304	-7.3866 (1.6640)	-0.4562	1.7254
(100,20)	0.5000	0.0000	0.0036 (0.2299)	0.0036	0.2299
	0.1240	-1.1551	-1.1957 (0.3028)	-0.0406	0.3055
	0.0047	-2.5989	-2.6949 (0.4952)	-0.0960	0.5044
	2.6e-05	-4.0427	-4.1941 (0.7178)	-0.1514	0.7336
	2.0e-08	-5.4865	-5.6933 (0.9497)	-0.2067	0.9719
	2.1e-12	-6.9304	-7.1925 (1.1854)	-0.2621	1.2140
(100,100)	0.5000	0.0000	-0.0024 (0.0998)	-0.0024	0.0998
	0.1240	-1.1551	-1.1640 (0.1279)	-0.0089	0.1282
	0.0047	-2.5989	-2.6160 (0.2039)	-0.0171	0.2046
	2.6e-05	-4.0427	-4.0681 (0.2931)	-0.0254	0.2942
	2.0e-08	-5.4865	-5.5201 (0.3864)	-0.0336	0.3879
	2.1e-12	-6.9304	-6.9722 (0.4815)	-0.0418	0.4833

**Table S12:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the Bayes estimator for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0147 (1.4158)	-0.0147	1.4159
	0.1240	-1.1551	-1.7090 (1.9791)	-0.5539	2.0551
	0.0047	-2.5989	-3.7772 (3.4930)	-1.1783	3.6864
	2.6e-05	-4.0427	-5.8375 (5.2056)	-1.7948	5.5064
	2.0e-08	-5.4865	-7.9007 (6.9669)	-2.4142	7.3734
	2.1e-12	-6.9304	-9.9670 (8.7480)	-3.0366	9.2600
(5,5)	0.5000	0.0000	-0.0011 (0.5290)	-0.0011	0.5290
	0.1240	-1.1551	-1.2708 (0.7376)	-0.1157	0.7466
	0.0047	-2.5989	-2.8446 (1.2681)	-0.2457	1.2917
	2.6e-05	-4.0427	-4.4173 (1.8678)	-0.3746	1.9050
	2.0e-08	-5.4865	-5.9906 (2.4865)	-0.5041	2.5371
	2.1e-12	-6.9304	-7.5646 (3.1131)	-0.6342	3.1770
(5,12)	0.5000	0.0000	-0.0011 (0.3072)	-0.0011	0.3072
	0.1240	-1.1551	-1.1812 (0.4009)	-0.0261	0.4017
	0.0047	-2.5989	-2.6515 (0.6577)	-0.0526	0.6598
	2.6e-05	-4.0427	-4.1214 (0.9565)	-0.0787	0.9597
	2.0e-08	-5.4865	-5.5916 (1.2677)	-0.1050	1.2720
	2.1e-12	-6.9304	-7.0619 (1.5841)	-0.1316	1.5895
(5,20)	0.5000	0.0000	-0.0011 (0.2410)	-0.0011	0.2410
	0.1240	-1.1551	-1.1583 (0.3012)	-0.0032	0.3012
	0.0047	-2.5989	-2.6017 (0.4775)	-0.0028	0.4775
	2.6e-05	-4.0427	-4.0447 (0.6874)	-0.0020	0.6874
	2.0e-08	-5.4865	-5.4878 (0.9079)	-0.0012	0.9079
	2.1e-12	-6.9304	-6.9310 (1.1328)	-0.0006	1.1328
(5,100)	0.5000	0.0000	0.0021 (0.1311)	0.0021	0.1311
	0.1240	-1.1551	-1.1464 (0.1567)	0.0087	0.1569
	0.0047	-2.5989	-2.5800 (0.2285)	0.0189	0.2293
	2.6e-05	-4.0427	-4.0130 (0.3170)	0.0297	0.3184
	2.0e-08	-5.4865	-5.4460 (0.4116)	0.0406	0.4136
	2.1e-12	-6.9304	-6.8790 (0.5088)	0.0514	0.5114
(100,2)	0.5000	0.0000	-0.0219 (1.3801)	-0.0219	1.3803
	0.1240	-1.1551	-1.8086 (1.9961)	-0.6535	2.1004
	0.0047	-2.5989	-3.9864 (3.5615)	-1.3875	3.8223
	2.6e-05	-4.0427	-6.1569 (5.3120)	-2.1142	5.7173
	2.0e-08	-5.4865	-8.3308 (7.1083)	-2.8442	7.6562
	2.1e-12	-6.9304	-10.5074 (8.9216)	-3.5770	9.6120
(100,5)	0.5000	0.0000	0.0032 (0.5371)	0.0032	0.5371
	0.1240	-1.1551	-1.2987 (0.7590)	-0.1436	0.7725
	0.0047	-2.5989	-2.9105 (1.3548)	-0.3116	1.3901
	2.6e-05	-4.0427	-4.5213 (2.0237)	-0.4786	2.0796
	2.0e-08	-5.4865	-6.1332 (2.7120)	-0.6466	2.7880
	2.1e-12	-6.9304	-7.7455 (3.4074)	-0.8151	3.5035
(100,12)	0.5000	0.0000	-0.0014 (0.2984)	-0.0014	0.2984
	0.1240	-1.1551	-1.2003 (0.4010)	-0.0452	0.4036
	0.0047	-2.5989	-2.6930 (0.6685)	-0.0941	0.6751
	2.6e-05	-4.0427	-4.1854 (0.9756)	-0.1427	0.9860
	2.0e-08	-5.4865	-5.6781 (1.2943)	-0.1916	1.3084
	2.1e-12	-6.9304	-7.1710 (1.6176)	-0.2406	1.6354
(100,20)	0.5000	0.0000	0.0036 (0.2267)	0.0036	0.2267
	0.1240	-1.1551	-1.1775 (0.2976)	-0.0224	0.2985
	0.0047	-2.5989	-2.6504 (0.4865)	-0.0515	0.4892
	2.6e-05	-4.0427	-4.1232 (0.7055)	-0.0804	0.7101
	2.0e-08	-5.4865	-5.5961 (0.9336)	-0.1096	0.9400
	2.1e-12	-6.9304	-7.0691 (1.1654)	-0.1388	1.1737
(100,100)	0.5000	0.0000	-0.0023 (0.0995)	-0.0023	0.0995
	0.1240	-1.1551	-1.1601 (0.1275)	-0.0051	0.1276
	0.0047	-2.5989	-2.6068 (0.2031)	-0.0079	0.2033
	2.6e-05	-4.0427	-4.0534 (0.2919)	-0.0107	0.2921
	2.0e-08	-5.4865	-5.5001 (0.3849)	-0.0135	0.3851
	2.1e-12	-6.9304	-6.9467 (0.4795)	-0.0164	0.4798

**Table S13:** The sample size, true  $R$ , true  $\text{probit}(R)$ , mean ( $\bar{R}$ ), standard deviation (SD), bias and root mean squared error (RMSE) of the non-parametric estimator for all sample size scenarios and selected  $R$  values.

$(n_x, n_y)$	$R$	$\text{probit}(R)$	$\bar{R}$ (SD)	bias	RMSE
(5,2)	0.5000	0.0000	-0.0261 (0.8929)	-0.0261	0.8933
	0.1240	-1.1551	-1.0225 (0.6026)	0.1326	0.6170
	0.0047	-2.5989	-1.3681 (0.1271)	1.2308	1.2373
	2.6e-05	-4.0427	-1.3830 (0.0000)	2.6597	2.6597
	2.0e-08	-5.4865	-1.3830 (0.0000)	4.1035	4.1035
	2.1e-12	-6.9304	-1.3830 (0.0000)	5.5474	5.5474
(5,5)	0.5000	0.0000	0.0026 (0.5799)	0.0026	0.5799
	0.1240	-1.1551	-1.1911 (0.5640)	-0.0361	0.5651
	0.0047	-2.5989	-1.7581 (0.1467)	0.8408	0.8535
	2.6e-05	-4.0427	-1.7862 (0.0000)	2.2566	2.2566
	2.0e-08	-5.4865	-1.7862 (0.0000)	3.7004	3.7004
	2.1e-12	-6.9304	-1.7862 (0.0000)	5.1442	5.1442
(5,12)	0.5000	0.0000	-0.0035 (0.3534)	-0.0035	0.3534
	0.1240	-1.1551	-1.2064 (0.4607)	-0.0513	0.4635
	0.0047	-2.5989	-2.0841 (0.1888)	0.5148	0.5483
	2.6e-05	-4.0427	-2.1408 (0.0145)	1.9019	1.9020
	2.0e-08	-5.4865	-2.1412 (0.0000)	3.3453	3.3453
	2.1e-12	-6.9304	-2.1412 (0.0000)	4.7892	4.7892
(5,20)	0.5000	0.0000	-0.0033 (0.2764)	-0.0033	0.2764
	0.1240	-1.1551	-1.1905 (0.3691)	-0.0354	0.3708
	0.0047	-2.5989	-2.2486 (0.2141)	0.3503	0.4106
	2.6e-05	-4.0427	-2.3335 (0.0143)	1.7092	1.7093
	2.0e-08	-5.4865	-2.3338 (0.0000)	3.1528	3.1528
	2.1e-12	-6.9304	-2.3338 (0.0000)	4.5966	4.5966
(5,100)	0.5000	0.0000	0.0028 (0.1459)	0.0028	0.1459
	0.1240	-1.1551	-1.1636 (0.1776)	-0.0085	0.1778
	0.0047	-2.5989	-2.5939 (0.3022)	0.0050	0.3023
	2.6e-05	-4.0427	-2.8775 (0.0293)	1.1652	1.1656
	2.0e-08	-5.4865	-2.8794 (0.0000)	2.6071	2.6071
	2.1e-12	-6.9304	-2.8794 (0.0000)	4.0509	4.0509
(100,2)	0.5000	0.0000	-0.0252 (1.4283)	-0.0252	1.4285
	0.1240	-1.1551	-1.8277 (1.0598)	-0.6726	1.2552
	0.0047	-2.5989	-2.5385 (0.2605)	0.0603	0.2674
	2.6e-05	-4.0427	-2.5790 (0.0134)	1.4638	1.4638
	2.0e-08	-5.4865	-2.5793 (0.0000)	2.9073	2.9073
	2.1e-12	-6.9304	-2.5793 (0.0000)	4.3511	4.3511
(100,5)	0.5000	0.0000	0.0011 (0.6613)	0.0011	0.6613
	0.1240	-1.1551	-1.5977 (0.9460)	-0.4426	1.0444
	0.0047	-2.5989	-2.8005 (0.3241)	-0.2017	0.3817
	2.6e-05	-4.0427	-2.8791 (0.0150)	1.1636	1.1637
	2.0e-08	-5.4865	-2.8794 (0.0000)	2.6071	2.6071
	2.1e-12	-6.9304	-2.8794 (0.0000)	4.0509	4.0509
(100,12)	0.5000	0.0000	0.0015 (0.3454)	0.0015	0.3454
	0.1240	-1.1551	-1.3374 (0.6226)	-0.1824	0.6488
	0.0047	-2.5989	-2.9741 (0.4364)	-0.3752	0.5755
	2.6e-05	-4.0427	-3.1432 (0.0336)	0.8995	0.9001
	2.0e-08	-5.4865	-3.1445 (0.0000)	2.3421	2.3421
	2.1e-12	-6.9304	-3.1445 (0.0000)	3.7859	3.7859
(100,20)	0.5000	0.0000	0.0048 (0.2616)	0.0048	0.2617
	0.1240	-1.1551	-1.2350 (0.4117)	-0.0799	0.4194
	0.0047	-2.5989	-3.0492 (0.4806)	-0.4503	0.6586
	2.6e-05	-4.0427	-3.2890 (0.0387)	0.7537	0.7547
	2.0e-08	-5.4865	-3.2908 (0.0000)	2.1957	2.1957
	2.1e-12	-6.9304	-3.2908 (0.0000)	3.6396	3.6396
(100,100)	0.5000	0.0000	-0.0019 (0.1153)	-0.0019	0.1153
	0.1240	-1.1551	-1.1692 (0.1483)	-0.0141	0.1490
	0.0047	-2.5989	-2.9163 (0.5665)	-0.3174	0.6493
	2.6e-05	-4.0427	-3.7098 (0.0852)	0.3329	0.3437
	2.0e-08	-5.4865	-3.7191 (0.0000)	1.7675	1.7675
	2.1e-12	-6.9304	-3.7191 (0.0000)	3.2113	3.2113